

UNC MATH 233 Study Notes

MULTIVARIABLE CALCULUS

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# Chapter 1

## Vector Algebra and Space Analytic Geometry

### 1.1 Vector Algebra

#### 1.1.1 Vector Basic Operations

**Definition:** A quantity with both magnitude and direction is called a **vector**.

Given two vectors  $\vec{a} = (a_x, a_y, a_z)$  and  $\vec{b} = (b_x, b_y, b_z)$ , and a scalar  $\lambda$ :

1. If two vectors  $\vec{a}$  and  $\vec{b}$  have equal magnitudes and directions, they are called **equal vectors**, denoted as  $\vec{a} = \vec{b}$ . This implies:

$$a_x = b_x, \quad a_y = b_y, \quad a_z = b_z$$

2. If two nonzero vectors  $\vec{a}$  and  $\vec{b}$  are either parallel or antiparallel, they are called **parallel vectors**, denoted as  $\vec{a} \parallel \vec{b}$ . This implies:

$$\vec{a} \parallel \vec{b} \Rightarrow \exists \lambda, \mu \neq 0, \quad \lambda \vec{a} + \mu \vec{b} = 0$$

Further, if  $\mu \neq 0$ , then:

$$\vec{b} = \lambda \vec{a} \Rightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$$

**Note:** A zero vector is parallel to any vector.

3. If  $k$  ( $\geq 3$ ) vectors lie on the same plane, they are said to be **coplanar vectors**.

4. A vector with the same magnitude but the opposite direction as  $\vec{a}$  is called the **negative vector**, denoted as  $-\vec{a}$ .
5. The addition and subtraction of vectors follow the parallelogram rule or triangle rule. Algebraically:

$$\vec{a} \pm \vec{b} = (a_x \pm b_x, a_y \pm b_y, a_z \pm b_z)$$

Scalar multiplication:

$$\lambda \vec{a} = (\lambda a_x, \lambda a_y, \lambda a_z)$$

### 1.1.2 Vector Products

Let  $\vec{a} = (a_x, a_y, a_z)$ ,  $\vec{b} = (b_x, b_y, b_z)$ , and  $\vec{c} = (c_x, c_y, c_z)$ , where  $\vec{a}, \vec{b}, \vec{c}$  are non-zero vectors.

#### Dot Product

Also called the inner product.

- $\vec{a} \cdot \vec{b} = (a_x, a_y, a_z) \cdot (b_x, b_y, b_z) = a_x b_x + a_y b_y + a_z b_z$ .
- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$ , so  $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \cdot \sqrt{b_x^2 + b_y^2 + b_z^2}}$ , where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ .
- The range of the angle between two vectors is  $0 \leq \theta(\vec{a}, \vec{b}) \leq \pi$
- $\vec{a} \perp \vec{b} \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = 0 \Leftrightarrow a_x b_x + a_y b_y + a_z b_z = 0$ .
- $Proj_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{b_x^2 + b_y^2 + b_z^2}}$ , called the **projection** of  $a$  onto  $b$ .

#### Cross Product

Also called the outer product.

- $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$ , where  $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$ . The direction is determined by the right-hand rule, and  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ .

- $\vec{a}/\vec{b} \Leftrightarrow \theta = 0 \Leftrightarrow \vec{a} \times \vec{b} = 0$  or  $\pi \Leftrightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$ .
- $\vec{a} \times \vec{a} = 0$ .

The cross product can also be computed by stacking two vectors together and duplicating the rightmost columns:

$\begin{bmatrix} a_x & a_y & a_z & a_x & a_y & a_z \\ b_x & b_y & b_z & b_x & b_y & b_z \end{bmatrix}$ . The first value of the cross product corresponds to the determinant of columns 2 and 3, the second to columns 3 and 4, and the third to columns 4 and 5. Columns 1 and 6 are not used.

### Triple Product

- $[\vec{a}\vec{b}\vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$ .
- Swapping two rows does not change the value:  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$ .
- Swapping one row changes the sign:  $\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{a} \cdot (\vec{c} \times \vec{b}) = -\vec{b} \cdot (\vec{a} \times \vec{c}) = -\vec{c} \cdot (\vec{b} \times \vec{a})$ .
- $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0 \Leftrightarrow$  the three vectors are coplanar.
- Four points  $A, B, C, D$  in space are coplanar if and only if  $(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = 0$ .
- The volume of a tetrahedron formed by vectors  $\vec{a}, \vec{b}, \vec{c}$  is:  $V_{\text{tetrahedron}} = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|$ .

### 1.1.3 Direction Angles and Direction Cosines of Vectors

- Basis vectors:  $\vec{i}, \vec{j}$ , and  $\vec{k}$  represent unit vectors along the  $x, y$ , and  $z$  axes, respectively, with a magnitude of 1.
- Any vector  $\vec{a} = (a_x, a_y, a_z)$  can be expressed as:  $\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$
- The components  $a_x, a_y$ , and  $a_z$  are called the projections of the vector  $\vec{a}$  onto the  $x, y$ , and  $z$  axes, respectively.

- The angles between the vector and the coordinate axes are denoted as  $\alpha$ ,  $\beta$ , and  $\gamma$ . These are known as the **direction angles** of the vector, and the corresponding cosines,  $\cos \alpha$ ,  $\cos \beta$ , and  $\cos \gamma$ , are referred to as the **direction cosines**. The following relationships hold:

$$\cos \alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

- A vector  $\vec{r} = (x, y, z)$  can also be written in terms of its magnitude and a unit vector in its direction:

$$\vec{r} = |\vec{r}|(\cos \alpha, \cos \beta, \cos \gamma) \quad \Rightarrow \quad \vec{r} = |\vec{r}|\vec{r}_0$$

where  $\vec{r}_0$  is the unit vector in the direction of  $\vec{r}$ .

## 1.2 Space Analytic Geometry

### 1.2.1 Equation of a Plane

Assume the normal vector of the plane is  $\vec{n} = (A, B, C)$ .

- General form:  $Ax + By + Cz + D = 0$ .
- Point-normal form:  $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ .
- Three-point form:  $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{vmatrix} = 0$  (the plane passes through three non-collinear points).
- Intercept form:  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$  (the plane passes through  $(a, 0, 0)$ ,  $(0, b, 0)$ ,  $(0, 0, c)$ ).

### 1.2.2 Equation of Lines

Assume the direction vector of the line is  $\vec{\tau} = (l, m, n)$ .

- General form:  $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \vec{n}_1 = (A_1, B_1, C_1) \\ A_2x + B_2y + C_2z + D_2 = 0, \vec{n}_2 = (A_2, B_2, C_2) \end{cases}$ , where  $\vec{n}_1$  is not parallel to  $\vec{n}_2$ . (The line is the intersection of two planes, and its direction vector  $\vec{\tau} = \vec{n}_1 \times \vec{n}_2$ )

- Point-direction form (standard form or symmetric form):  $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$   
(a point on the line is proportional to the direction vector).
- Parametric form:  $\begin{cases} x = x_0 + lt \\ y = y_0 + mt \\ z = z_0 + nt \end{cases}$ , where  $M(x_0, y_0, z_0)$  is a known point on the line,  
and  $t$  is the parameter.
- Two-point form:  $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$  (the line passes through two distinct points).

### 1.2.3 Positional Relationships

#### Relationship Between Lines

Let  $\vec{\tau}_1 = (l_1, m_1, n_1)$  and  $\vec{\tau}_2 = (l_2, m_2, n_2)$  be the direction vectors of lines  $L_1$  and  $L_2$ , respectively.

- $L_1 \perp L_2 \Leftrightarrow \vec{\tau}_1 \perp \vec{\tau}_2 \Leftrightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ .
- $L_1 // L_2 \Leftrightarrow \vec{\tau}_1 // \vec{\tau}_2 \Leftrightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ .

#### Relationship Between Planes

Let  $\pi_1$  and  $\pi_2$  have normal vectors  $\vec{n}_1 = (A_1, B_1, C_1)$  and  $\vec{n}_2 = (A_2, B_2, C_2)$ , respectively.

- $\pi_1 \perp \pi_2 \Leftrightarrow \vec{n}_1 \perp \vec{n}_2 \Leftrightarrow A_1 A_2 + B_1 B_2 + C_1 C_2 = 0$ .
- $\pi_1 // \pi_2 \Leftrightarrow \vec{n}_1 // \vec{n}_2 \Leftrightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$ .

#### Relationship Between a Line and a Plane

Let the direction vector of the line  $L$  be  $\vec{\tau} = (l, m, n)$ , and the normal vector of the plane  $\pi$  be  $\vec{n} = (A, B, C)$ .

- $L \perp \pi \Leftrightarrow \vec{\tau} // \vec{n} \Leftrightarrow \frac{A}{l} = \frac{B}{m} = \frac{C}{n}$ .
- $L // \pi \Leftrightarrow \vec{\tau} \perp \vec{n} \Leftrightarrow Al + Bm + Cn = 0$ .

## Distance

Distance formulas:

- Distance from a point to a line in 2D: The distance from point  $P_0(x_0, y_0, z_0)$  to the line  $Ax + By + C = 0$  is  $d = \frac{|Ax_0 + By_0 + Cz_0|}{\sqrt{A^2 + B^2}}$ .
- Distance from a point to a plane in 3D: The distance from point  $P_0(x_0, y_0, z_0)$  to the plane  $Ax + By + Cz + D = 0$  is  $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ .
- Distance between two parallel lines in 2D: The distance between lines  $Ax + By + C_1 = 0$  and  $Ax + By + C_2 = 0$  is  $d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}$ .
- Angle between two non-parallel lines in 2D: The angle  $\theta$  between lines  $A_1x + B_1y + C_1 = 0$  and  $A_2x + B_2y + C_2 = 0$  is given by  $\cos \theta = \frac{|A_1A_2 + B_1B_2|}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}$  ( $\theta \in [0, \frac{\pi}{2}]$ ).
- Distance between two parallel planes in 3D: The distance between planes  $Ax + By + Cz + D_1 = 0$  and  $Ax + By + Cz + D_2 = 0$  is  $d = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}$  (choose any point on one plane and calculate its distance to the other plane).

Distance from a point to a line in 3D (given a line  $L$  in general form and a point  $M_1$ ):

1. Derive the normal vectors  $\vec{\xi}_1$  and  $\vec{\xi}_2$  of two planes from the general form of  $L$ .
2. Compute the direction vector  $\vec{S} = \vec{\xi}_1 \times \vec{\xi}_2$  of the line  $L$  using the cross product.
3. Find any point  $M_0$  on  $L$ , then compute the vector  $\overrightarrow{M_0M_1}$ , and calculate the cross product  $\overrightarrow{M_0M_1} \times \vec{S}$ . The magnitude  $|\overrightarrow{M_0M_1} \times \vec{S}|$  represents twice the area of the triangle formed by the three points.
4. Compute the magnitude of the direction vector  $\vec{S}$ , so the distance  $d$  from  $M_1$  to  $L$  can be expressed as  $d \cdot |\vec{S}| = 2S_{\triangle M_0M_1S}$ .
5. Solve for  $d$ :  $d = \frac{|\overrightarrow{M_0M_1} \times \vec{S}|}{|\vec{S}|}$ .

### 1.2.4 Space Curves

The tangent vector at a point on a space curve is equal to the derivative at that point.

**Representation**

- General form:  $\Gamma : \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$ , representing the intersection of two surfaces.
- Parametric equations:  $\Gamma : \begin{cases} x = \phi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases}$ , where  $t \in [\alpha, \beta]$ .

**Projection of a Space Curve onto a Coordinate Plane**

To find the projection of a curve  $\Omega$  onto the  $xOy$  plane, eliminate  $z$  from the equations of  $\Gamma : \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$  to obtain  $\varphi(x, y) = 0$ . The projected curve  $\Omega$  on the  $xOy$  plane is contained in  $\begin{cases} \varphi(x, y) = 0 \\ z = 0 \end{cases}$ .

**1.2.5 Space Surfaces****Equations of Surfaces**

Implicit form:  $F(x, y, z) = 0$ , explicit form:  $z = z(x, y)$ .

**Quadratic Surfaces**

- Sphere:  $x^2 + y^2 + z^2 = r^2$ .
- Ellipsoid:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ .
- Hyperboloid of one sheet:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ .
- Hyperboloid of two sheets:  $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ .
- Elliptic paraboloid:  $\frac{x^2}{2p} + \frac{y^2}{2q} = z$  ( $p, q > 0$ ). (Frequently tested: rotational paraboloid  $x^2 + y^2 = z$ )
- Elliptic cone:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ . (Frequently tested: rotational cone  $z = \sqrt{x^2 + y^2}$ )

- Hyperbolic paraboloid (saddle surface):  $-\frac{x^2}{2p} + \frac{y^2}{2q} = z$  ( $p, q > 0$ ). (Frequently tested:  $z = xy$ )

## Cylindrical Surfaces

In space analytic geometry, a surface lacking one variable is generally considered a cylindrical surface.

It is formed by a moving line parallel to a fixed curve.

- Elliptic cylinder:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  (if  $a = b$ , it becomes a circular cylinder).
- Hyperbolic cylinder:  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .
- Parabolic cylinder:  $y = ax^2$ .

## Surfaces of Revolution

A surface generated by revolving a curve around an axis. Replace one variable with the square root of the sum of the squares of the other two variables.

For example, if  $f(x, y) = 0$  is revolved around the  $x$ -axis, it becomes  $f(x, \pm\sqrt{y^2 + z^2})$ .

It is formed by revolving a curve  $\Gamma$  around a fixed line.

Given a line  $L : \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$  with direction vector  $\vec{\tau}(l, m, n)$  and a point  $P_0(x_0, y_0, z_0)$  on it.

Now given a curve  $\Gamma : \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$ .

Find a point  $P_1(x_1, y_1, z_1)$  on  $\Gamma$ , and revolve  $P_1$  around  $L$  to form a circle of latitude. Any point  $P(x, y, z)$  on this circle is a point on the surface of revolution.

Since  $P_1$  lies on  $\Gamma$ ,  $F(x_1, y_1, z_1) = 0$  and  $G(x_1, y_1, z_1) = 0$ .

The distance from  $P_1$  to  $P_0$  on  $L$  is equal to the distance from  $P$  to  $P_0$ , i.e.,  $|\overrightarrow{P_1P_0}| = |\overrightarrow{PP_0}|$ , which gives  $(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$ .

Since the plane of the circle is perpendicular to the direction vector  $\vec{\tau}$  of  $L$ , and  $P_1P$  lies in the plane, the vector  $\overrightarrow{P_1P}$  is perpendicular to  $\vec{\tau}$ , i.e.,  $l(x - x_1) + m(y - y_1) + n(z - z_1) = 0$ .

To obtain the equation of the surface of revolution, eliminate  $x_1, y_1, z_1$  to get  $H(x, y, z) = 0$ .

**Example:** Find the equation of the surface generated by revolving the curve  $L$  :

$$\begin{cases} x - y + 2z - 1 = 0 \\ x - 3y - 2z + 1 = 0 \end{cases} \quad \text{around the } y\text{-axis.}$$

Solution:

Let  $P_1(x_1, y_1, z_1)$  lie on the curve, so  $x_1 - y_1 + 2z_1 - 1 = 0$  and  $x_1 - 3y_1 - 2z_1 + 1 = 0$ .

For any point  $P(x, y, z)$ , the distance from  $P_0(0, 0, 0)$  to  $P$  is equal to the distance from  $P_0$  to  $P_1$ , hence  $x_1^2 + y_1^2 + z_1^2 = x^2 + y^2 + z^2$ .

Since the line segments are perpendicular to the  $y$ -axis,  $\overrightarrow{P_1P} \perp (0, 1, 0)$ , which implies  $y = y_1$ .

Eliminate  $x_1, y_1, z_1$ : given  $y = y_1$ , we have  $x_1^2 + z_1^2 = x^2 + z^2$ .

Solving the line equations gives  $x_1 = 2y$  and  $z_1 = \frac{1}{2}(1 - y)$ .

Substitute these into  $x^2 + z^2 = (2y)^2 + \frac{1}{4}(1 - y)^2$ , resulting in the equation  $x^2 - \frac{17}{4}y^2 + z^2 + \frac{y}{2} - \frac{1}{4} = 0$ .

## 1.3 Spherical and Cylindrical Coordinates

### 1.3.1 Relationship Between Spherical and Cartesian Coordinates

The conversion formulas from spherical coordinates  $(r, \theta, \phi)$  to Cartesian coordinates  $(x, y, z)$  are given by:

$$\begin{cases} x = r \sin \phi \cos \theta & 0 \leq r < +\infty \\ y = r \sin \phi \sin \theta & 0 \leq \phi \leq \pi \\ z = r \cos \phi & 0 \leq \theta < 2\pi \end{cases}$$

The conversion formulas from Cartesian coordinates  $(x, y, z)$  to spherical coordinates  $(r, \theta, \phi)$  are:

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \tan \theta = \frac{y}{x} \\ \phi = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{cases} \quad (-\infty < x, y, z < +\infty)$$

**Note:** The range of  $\theta$  depends on the quadrant in which  $(x, y)$  lies, and it is determined by the inverse tangent function.

### 1.3.2 Relationship Between Cylindrical and Cartesian Coordinates

The conversion formulas from cylindrical coordinates  $(\rho, \theta, z)$  to Cartesian coordinates  $(x, y, z)$  are given by:

$$\begin{cases} x = \rho \cos \theta & 0 \leq \rho < +\infty \\ y = \rho \sin \theta & 0 \leq \theta < 2\pi \\ z = z & -\infty < z < +\infty \end{cases}$$

The conversion formulas from Cartesian coordinates  $(x, y, z)$  to cylindrical coordinates  $(\rho, \theta, z)$  are:

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \tan \theta = \frac{y}{x} \\ z = z \end{cases} \quad (-\infty < x, y, z < +\infty)$$

**Note:** The range of  $\theta$  depends on the quadrant in which  $(x, y)$  lies, and it is determined by the inverse tangent function.

# Chapter 2

## Vector-Valued Functions

### 2.1 Basic Concepts of Vector-Valued Functions

#### 2.1.1 What Is a Vector-Valued Function

A **scalar function** is a mapping from a defined domain to the range of values, where both the domain and range are one-dimensional. A **vector-valued function** refers to a function whose values are vectors. When the domain is a one-dimensional scalar variable, a vector-valued function represents a mapping from  $\mathbb{R}$  to  $\mathbb{R}^n$ .

For example:

- **Two-dimensional vector-valued function:**

$$\vec{r}(t) = \overrightarrow{OP} = (f(t), g(t)) = f(t)\vec{i} + g(t)\vec{j}$$

- **Three-dimensional vector-valued function:**

$$\vec{r}(t) = (f(t), g(t), h(t)) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

The components  $f(t)$ ,  $g(t)$ , and  $h(t)$  are functions of  $t$  in the plane or space, respectively. The corresponding representations are parametric equations of a plane curve or a space curve.

#### 2.1.2 Limits and Continuity of Vector-Valued Functions

Consider a two-dimensional vector-valued function:

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$$

For a three-dimensional vector-valued function:

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

Assume that  $f(t)$  and  $g(t)$  are defined in a neighborhood of  $t_0$ . If both of the following limits exist:

$$\lim_{t \rightarrow t_0} f(t) = a, \quad \lim_{t \rightarrow t_0} g(t) = b$$

Then the limit of the vector-valued function  $\vec{r}(t)$  at  $t_0$  exists, and its limit is given by:

$$\lim_{t \rightarrow t_0} \vec{r}(t) = a\vec{i} + b\vec{j}$$

If the vector-valued function  $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$  is defined in a neighborhood of  $t_0$ , and:

$$\lim_{t \rightarrow t_0} \vec{r}(t) = \vec{r}(t_0)$$

then  $\vec{r}(t)$  is continuous at  $t_0$ .

**Note:** For the two-dimensional vector-valued function  $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$  to be continuous at  $t_0$ , a necessary and sufficient condition is that the scalar functions  $f(t)$  and  $g(t)$  are both continuous at  $t_0$ .

## 2.2 Differentiate Vector-Valued Functions

### 2.2.1 Derivatives and Differentiation of Vector-Valued Functions

Suppose a vector-valued function  $\vec{r} = \vec{r}(t)$  is defined in a neighborhood of  $t$ , and the limit

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

exists. Then the vector-valued function  $\vec{r}$  is said to be differentiable at  $t$ , and the limit is called the derivative of the vector-valued function at  $t$ , denoted by  $\vec{r}'(t)$  or  $\frac{d\vec{r}(t)}{dt}$ .

**Note 1:**  $\vec{r}'(t)$  is also a vector-valued function. If the vector-valued function  $\vec{r}(t)$  is differentiable at  $t$ , then it must be continuous at  $t$ .

**Note 2:** Let the three-dimensional vector-valued function be  $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ , where each component function is differentiable at  $t$ . Then  $\vec{r}(t)$  is differentiable at  $t$ , and:

$$\vec{r}'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

$$\vec{r}''(t) = f''(t)\vec{i} + g''(t)\vec{j} + h''(t)\vec{k}$$

**Note 3:** If a vector-valued function  $\vec{r}(t)$  is continuous on an interval  $I$  and satisfies  $\vec{r}'(t) \neq 0$  on  $I$ , then  $\vec{r}(t)$  is said to be smooth on  $I$ . A curve composed of multiple smooth segments is called a piecewise smooth curve.

**Note 4:** Differential of a vector-valued function:

$$d\vec{r} = df(t)\vec{i} + dg(t)\vec{j} = f'(t)dt\vec{i} + g'(t)dt\vec{j}$$

$$d\vec{r} = df(t)\vec{i} + dg(t)\vec{j} + dh(t)\vec{k} = f'(t)dt\vec{i} + g'(t)dt\vec{j} + h'(t)dt\vec{k}$$

$$= (f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k})dt$$

### 2.2.2 Rules for Differentiating Vector-Valued Functions

Let  $u(t)$  and  $v(t)$  be differentiable vector-valued functions,  $f(t)$  be a differentiable scalar function,  $C$  be a constant vector (i.e., each component of  $C$  is constant), and  $k$  be a constant. Then the following rules hold:

1.  $\frac{d}{dt}C = 0$
2.  $\frac{d}{dt}[u(t) \pm v(t)] = u'(t) \pm v'(t)$
3.  $\frac{d}{dt}[ku(t)] = ku'(t)$
4.  $\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$
5.  $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$
6.  $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$
7. **Chain Rule:** Let  $u(s)$  be a differentiable vector-valued function and  $s = f(t)$  be a differentiable scalar function. Then:

$$\frac{du(s)}{dt} = \frac{du}{ds} \cdot \frac{ds}{dt} = u'(s)f'(t) = u'(f(t))f'(t)$$

### 2.2.3 Tangent Line and Normal Plane of a Space Curve

Given a space curve  $C : x = f(t), y = g(t), z = h(t)$ , the **tangent vector** at point  $t = t_0$  is:

$$\vec{T}(t_0) = (f'(t_0), g'(t_0), h'(t_0))$$

**Note:** This computation yields the direction of the tangent line where the parameter  $t$  increases.

The equation of the tangent line of the space curve  $C$  at point  $P_0 (f(t_0), g(t_0), h(t_0))$  is:

$$\frac{x - f(t_0)}{f'(t_0)} = \frac{y - g(t_0)}{g'(t_0)} = \frac{z - h(t_0)}{h'(t_0)}$$

A plane perpendicular to the tangent vector  $\vec{T}$  at point  $P_0$  is called the normal plane of the space curve  $C$ . Its equation is:

$$f'(t_0)(x - f(t_0)) + g'(t_0)(y - g(t_0)) + h'(t_0)(z - h(t_0)) = 0$$

## 2.3 Integral of Vector-Valued Functions

### 2.3.1 Indefinite Integral of Vector-Valued Functions

#### Antiderivative of a Vector-Valued Function

Let the vector-valued function  $\vec{r} = \vec{r}(t)$  be defined on an interval  $I$ . If there exists a differentiable vector-valued function  $\vec{R}(t)$  such that for any  $t \in I$ :

$$\vec{R}'(t) = \vec{r}(t)$$

then  $\vec{R}(t)$  is called an **antiderivative** of  $\vec{r}(t)$  on  $I$ .

**Note 1:** If  $\vec{R}(t)$  is an antiderivative of  $\vec{r}(t)$  on  $I$ , then each component of  $\vec{R}(t)$  is also an antiderivative of the corresponding component of  $\vec{r}(t)$  on  $I$ .

**Note 2:** The general form of any antiderivative of  $\vec{r}(t)$  on  $I$  is  $\vec{R}(t) + \vec{C}$ , where  $\vec{C}$  is a constant vector.

**Note 3:** If the vector-valued function  $\vec{r}(t)$  is continuous on  $I$ , then it has at least one antiderivative on  $I$ .

#### Indefinite Integral of a Vector-Valued Function

If  $\vec{R}(t)$  is an antiderivative of  $\vec{r}(t)$  on  $I$ , then the indefinite integral of  $\vec{r}(t)$  on  $I$  is given by:

$$\int \vec{r}(t) dt = \vec{R}(t) + \vec{C}$$

### 2.3.2 Definite Integral of Vector-Valued Functions

Let the three-dimensional vector-valued function  $\vec{r}(t) = (f(t), g(t), h(t))$  be continuous on the interval  $[a, b]$ . The **definite integral** of  $\vec{r}(t)$  on  $[a, b]$  is defined as:

$$\int_a^b \vec{r}(t) dt = \left( \int_a^b f(t) dt \right) \vec{i} + \left( \int_a^b g(t) dt \right) \vec{j} + \left( \int_a^b h(t) dt \right) \vec{k}$$

The **Fundamental Theorem of Calculus for Vector-Valued Functions** states that:

If the vector-valued function  $\vec{r} = \vec{r}(t)$  is continuous on  $[a, b]$  and  $\vec{R}(t)$  is an antiderivative of  $\vec{r}(t)$  on  $[a, b]$ , then:

$$\int_a^b \vec{r}(t) dt = \vec{R}(t) \Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

## 2.4 Application of Vector-Valued Functions

### 2.4.1 Derivative of the Dot Product

Given two time-dependent vector functions  $\vec{a}(t)$  and  $\vec{b}(t)$ , the derivative of their dot product  $\langle \vec{a}(t), \vec{b}(t) \rangle$  can be written as:

$$\frac{d}{dt} \langle \vec{a}(t), \vec{b}(t) \rangle = \frac{d}{dt} \sum_{i=1}^n a_i(t) b_i(t)$$

This is because the dot product is the sum of the products of each component. Using the product rule (Leibniz rule), we can differentiate each term  $a_i(t) b_i(t)$ :

$$\frac{d}{dt} (a_i(t) b_i(t)) = \frac{da_i(t)}{dt} b_i(t) + a_i(t) \frac{db_i(t)}{dt}$$

Summing the derivatives of all components, we get:

$$\frac{d}{dt} \langle \vec{a}(t), \vec{b}(t) \rangle = \sum_{i=1}^n \left( \frac{da_i(t)}{dt} b_i(t) + a_i(t) \frac{db_i(t)}{dt} \right)$$

This can be rewritten in vector form as:

$$\frac{d}{dt} \langle \vec{a}(t), \vec{b}(t) \rangle = \left\langle \frac{d\vec{a}(t)}{dt}, \vec{b}(t) \right\rangle + \left\langle \vec{a}(t), \frac{d\vec{b}(t)}{dt} \right\rangle$$

This is the general formula for the derivative of the dot product.

### 2.4.2 Arc Length of a Curve

For a vector-valued function  $\vec{r}(t)$  in  $n$ -dimensional space, where:

$$\vec{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_n(t) \end{pmatrix}$$

the arc length  $L$  of the curve from  $t = a$  to  $t = b$  is defined as:

$$L = \int_a^b \|\vec{r}'(t)\| dt = \int_a^b \sqrt{\begin{pmatrix} r'_1(t) \\ r'_2(t) \\ \vdots \\ r'_n(t) \end{pmatrix}^T \begin{pmatrix} r'_1(t) \\ r'_2(t) \\ \vdots \\ r'_n(t) \end{pmatrix}} dt$$

This simplifies to:

$$L = \int_a^b \sqrt{r'_1(t)^2 + r'_2(t)^2 + \cdots + r'_n(t)^2} dt$$

### 2.4.3 Graphs and Level Curves of Functions of Several Variables

#### Definition of Level Curves

Given a scalar function  $f(x, y)$  of two variables, a level curve is the set of points  $(x, y)$  in the plane where the function has a constant value  $c$ . Mathematically, the level curve for a given  $c$  is:

$$f(x, y) = c$$

Level curves represent the contours of the function on the  $xy$ -plane and help visualize how the function behaves.

#### Graphs of Functions of Two Variables

The graph of a function  $f(x, y)$  of two variables is a surface in three-dimensional space given by:

$$z = f(x, y)$$

This surface consists of all points  $(x, y, z)$  where  $z$  is determined by applying the function  $f$  to  $(x, y)$ .

# Chapter 3

## Multivariable Differentiation

### 3.1 Partial Derivatives

#### 3.1.1 Definition of Partial Derivatives

When a function involves two or three variables, calculating a limit requires multiple variables to approach simultaneously, meaning that multiple variables change simultaneously. To simplify the computation, we assume that only one variable changes while others remain fixed. This reduces the problem to a single-variable case, where differentiation with respect to just one variable yields the **partial derivative**.

**Definition:** Let  $z = f(x, y)$  be a function defined in a neighborhood of the point  $(x_0, y_0)$ . If the following limit exists:

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

then this limit is called the **partial derivative** of  $z = f(x, y)$  with respect to  $x$  at the point  $(x_0, y_0)$  and is denoted by:

$$\left. \frac{\partial z}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}}, \quad \left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}}, \quad z'_x(x_0, y_0), \quad f'_x(x_0, y_0)$$

The partial derivative with respect to  $x$  can be written as:

$$f'_x(x_0, y_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

Similarly, the partial derivative with respect to  $y$  is defined as:

$$f'_y(x_0, y_0) = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = \lim_{y \rightarrow y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

**Definition:** If the partial derivatives  $f'_x(x, y)$  and  $f'_y(x, y)$  exist in a region  $D$ , and they themselves have partial derivatives, then these partial derivatives are called the **second-order partial derivatives** of  $z = f(x, y)$ . Depending on the order of differentiation, the following four second-order partial derivatives exist:

$$\begin{aligned}\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) &= \frac{\partial^2 z}{\partial x^2} = f''_{xx}(x, y), & \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) &= \frac{\partial^2 z}{\partial y^2} = f''_{yy}(x, y) \\ \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) &= \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x, y), & \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) &= \frac{\partial^2 z}{\partial y \partial x} = f''_{yx}(x, y)\end{aligned}$$

Here,  $f''_{xy}(x, y)$  and  $f''_{yx}(x, y)$  are called **mixed partial derivatives**. Second-order and higher-order partial derivatives are collectively referred to as **higher-order partial derivatives**.

### 3.1.2 Total Differential

**Definition:** Let  $z = f(x, y)$  be a function defined at a point  $(x, y)$ . If the total increment of  $z$ ,

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

can be expressed as:

$$\Delta z = A\Delta x + B\Delta y + o(\rho), \quad \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

where  $A$  and  $B$  depend only on  $(x, y)$  and not on  $\Delta x$  or  $\Delta y$ , then  $z = f(x, y)$  is said to be differentiable at the point  $(x, y)$ , and  $A\Delta x + B\Delta y$  is called the **total differential** of  $z = f(x, y)$  at the point  $(x, y)$ , denoted by  $dz$ .

The total differential  $dz$  is given by:

$$dz = A\Delta x + B\Delta y = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

#### Steps to Determine Differentiability:

1. Compute the total increment  $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ .
2. Compute the linear increment  $A\Delta x + B\Delta y$ , where  $A = f'_x(x_0, y_0)$  and  $B = f'_y(x_0, y_0)$ .
3. Evaluate the limit:

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta z - (A\Delta x + B\Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

If the limit equals 0, then  $z = f(x, y)$  is differentiable at the point  $(x_0, y_0)$ ; otherwise, it is not differentiable.

## 3.2 Limits and Continuity of Multivariable Functions

### 3.2.1 Limits of Multivariable Functions

For single-variable functions, limits can be evaluated by approaching the point from both sides along the x-axis. However, for multivariable functions, since the neighborhood around a point is multi-directional, it is impossible to evaluate the limit by approaching from only two directions.

**Definition:** Let  $f(x, y)$  be a function defined in a punctured neighborhood of  $(x_0, y_0)$ . If  $f(x, y)$  approaches a value  $A$  as  $(x, y)$  approaches  $(x_0, y_0)$  in any manner, then the limit exists and is denoted by:

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = A$$

### 3.2.2 Continuity of Multivariable Functions

**Definition:** A function  $f(x, y)$  is said to be **continuous** at a point  $(x_0, y_0)$  if:

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = f(x_0, y_0)$$

If the function is not continuous, the type of discontinuity is not discussed.

### 3.2.3 Continuity of Partial Derivatives

For  $z = f(x, y)$ , the steps to determine whether its partial derivatives are continuous at a specific point  $(x_0, y_0)$  are as follows:

1. Use the definition method to find  $f'_x(x_0, y_0)$  and  $f'_y(x_0, y_0)$ . (Calculate the partial derivatives at the given point)
2. Use the formula method to find  $f'_x(x, y)$  and  $f'_y(x, y)$ . (Obtain the partial derivative functions)
3. Compute  $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f'_x(x, y)$  and  $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f'_y(x, y)$ . (Take the limit of the partial derivative functions)
4. If  $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f'_x(x, y) = f'_x(x_0, y_0)$  and  $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f'_y(x, y) = f'_y(x_0, y_0)$  hold, then the partial derivatives are continuous; otherwise, they are discontinuous.

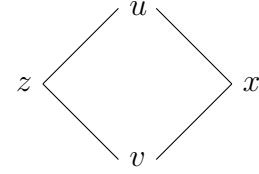
### 3.3 Differentiation Rules for Multivariable Functions

#### 3.3.1 Chain Rule for Differentiation

The chain rule for multivariable functions primarily deals with the differentiation of explicit functions.

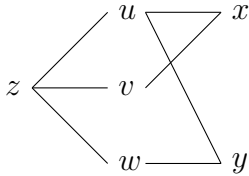
The chain rule for multivariable functions is analogous to that for single-variable functions. It involves tracing paths from the dependent variable through intermediate variables to the independent variables. Each path corresponds to one term in the final derivative, and the total number of terms is determined by the number of distinct paths from the dependent variable to all independent variables. Each path consists of several segments. If there are  $n$  layers of intermediate variables, then there are  $n + 1$  segments in each path. The segments between variables are multiplied together, where partial derivatives ( $\partial$ ) are used for variables with multiple paths, and total derivatives ( $d$ ) are used for variables with only one path.

For the dependent variable  $z$ , there are two distinct paths leading to  $x$ , resulting in two additive terms. Each path consists of two segments. Since  $z$  depends on two intermediate variables  $u$  and  $v$ , the partial derivatives are  $\frac{\partial z}{\partial u}$  and  $\frac{\partial z}{\partial v}$ . Both  $u$  and  $v$  have a direct path to  $x$ , so total derivatives are used:  $\frac{du}{dx}$  and  $\frac{dv}{dx}$ . Multiplying the terms along each path yields:



$$\frac{dz}{dx} = \frac{\partial z}{\partial u} \frac{du}{dx} + \frac{\partial z}{\partial v} \frac{dv}{dx}.$$

Since there are multiple paths from the dependent variable  $z$  to the independent variables  $x$  and  $y$ , they need to be analyzed separately.



For  $x$ , there are two paths:  $z \rightarrow u \rightarrow x$  and  $z \rightarrow v \rightarrow x$ . Since  $v$  only connects directly to  $x$ , it uses the total derivative:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{dv}{dx}.$$

Similarly, for  $y$ , the paths are  $z \rightarrow u \rightarrow y$  and  $z \rightarrow w \rightarrow y$ . Since  $w$  has only one direct path to  $y$ , it uses the total derivative:

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial w} \frac{dw}{dy}.$$

Regardless of which variable  $z$  is differentiated with respect to or how many derivatives are taken, the resulting function retains the same composite structure as the original.

**Example:** Given  $z = f(e^x \sin y, x^2 + y^2)$ , where  $f$  has continuous second-order partial derivatives, find  $\frac{\partial^2 z}{\partial x \partial y}$ .

Solution:

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right), \quad \therefore \frac{\partial z}{\partial x} = f'_1 \cdot e^x \sin y + f'_2 \cdot 2x.$$

Here, the first intermediate variable is  $f_1 = e^x \sin y$  (denoted as  $u$  earlier), and the second intermediate variable is  $f_2 = x^2 + y^2$  (denoted as  $v$  earlier). Let  $f'_1$  and  $f'_2$  represent the partial derivatives of  $f$  with respect to  $u$  and  $v$ , respectively.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial(f'_1 \cdot e^x \sin y)}{\partial y} + \frac{\partial(f'_2 \cdot 2x)}{\partial y}.$$

Expanding the terms:

$$\frac{\partial(f'_1 \cdot e^x \sin y)}{\partial y} = \frac{\partial f'_1}{\partial y} \cdot e^x \sin y + f'_1 \cdot e^x \cos y.$$

Thus, the challenge lies in finding  $\frac{\partial f'_1}{\partial y}$ .

The differentiation paths for  $f'_1$  are  $f'_1 \rightarrow 1 \rightarrow y$  and  $f'_1 \rightarrow 2 \rightarrow y$ :

$$= (f''_{11} e^x \cos y + f''_{12} 2y) \cdot e^x \sin y + f'_1 \cdot e^x \cos y.$$

For the second term:

$$\frac{\partial(f'_2 \cdot 2x)}{\partial y} = 2x \frac{\partial f'_2}{\partial y},$$

where the paths for  $f'_2$  are  $f'_2 \rightarrow 1 \rightarrow y$  and  $f'_2 \rightarrow 2 \rightarrow y$ :

$$= 2x(f''_{21} \cdot e^x \cos y + f''_{22} \cdot 2y).$$

Therefore,

$$\frac{\partial^2 z}{\partial x \partial y} = (f''_{11} e^x \cos y + f''_{12} 2y) \cdot e^x \sin y + f'_1 \cdot e^x \cos y + 2x(f''_{21} \cdot e^x \cos y + f''_{22} \cdot 2y).$$

If  $f$  has continuous second-order partial derivatives, then  $f''_{12} = f''_{21}$ .

Simplifying the expression:

$$= f'_1 e^x \cos y + f''_{11} e^{2x} \sin y \cos y + 2e^x f''_{12} (y \sin y + x \cos y) + 4f''_{22} xy.$$

### 3.3.2 Implicit Function Theorem

The main focus here is differentiating implicit functions. The biggest challenge with implicit functions is that the variables are intertwined, whereas in explicit functions obtained through differentiation, the variables are independent.

If, for each  $x \in D$ , there is a unique corresponding value of  $y$ , the defined function is called a **single-valued function**. If a given rule produces multiple possible values of  $y$  corresponding to a single  $x$ , it is not a function but a **multi-valued function**.

As long as the conditions on the domain are satisfied, a function in the form  $y = f(x)$  is an **explicit function**, such as  $y = \sin x$ . A function defined by an equation of the form  $F(x, y) = 0$  is an **implicit function**, such as  $x + y^3 - 1 = 0$ , which can be explicitly expressed as  $y = \sqrt[3]{1 - x}$ .

**Definition:** Suppose the function  $F(x, y)$  has continuous partial derivatives in a neighborhood of the point  $P(x_0, y_0)$  and satisfies  $F(x_0, y_0) = 0$  and  $F'_y(x_0, y_0) \neq 0$ . Then, there exists a unique continuous function  $y = f(x)$  with continuous derivatives in a neighborhood of  $(x_0, y_0)$  such that  $y_0 = f(x_0)$ .

**Definition:** The formula for differentiating a two-variable implicit function is:

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y}.$$

Since  $y = y(x)$ , differentiating  $F(x, y)$  with respect to  $x$  yields:

$$F'_x \cdot 1 + F'_y \cdot \frac{dy}{dx} = 0,$$

which leads to the implicit differentiation formula. The condition  $F'_y(x_0, y_0) \neq 0$  is crucial for the theorem to hold.

For example, given the equation of a circle  $F(x, y) = x^2 + y^2 - 1 = 0$ :

$$F'_x = 2x, \quad F'_y = 2y, \quad F(0, 1) = 0, \quad F'_y(0, 1) = 2 \neq 0.$$

Thus, in the neighborhoods of  $(0, 1)$  and  $(0, -1)$ , the function is single-valued and determines a unique implicit function with continuous derivatives. However, in the neighborhood of  $(\pm 1, 0)$ , the function does not exist since the tangent line is vertical.

**Definition:** Suppose the function  $F(x, y, z)$  has continuous partial derivatives in a neighborhood of the point  $P(x_0, y_0, z_0)$  and satisfies  $F(x_0, y_0, z_0) = 0$  and  $F'_z(x_0, y_0, z_0) \neq 0$ . Then, there exists a unique continuous function  $z = f(x, y)$  with continuous derivatives in a neighborhood of  $(x_0, y_0, z_0)$  such that  $z_0 = f(x_0, y_0)$ .

**Definition:** The formulas for differentiating a three-variable implicit function are:

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z}, \quad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}.$$

Since  $x$  is a function of  $x$ ,  $y$  is a function of  $y$ , and  $z$  is a function of both  $x$  and  $y$ , differentiating  $F(x, y, z)$  with respect to  $x$  yields:

$$F'_x \cdot 1 + F'_z \cdot \frac{\partial z}{\partial x} = 0,$$

which leads to the formula  $\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z}$ . Similarly, differentiating with respect to  $y$  yields the formula for  $\frac{\partial z}{\partial y}$ .

**Example:** Suppose  $z = f(x, y)$  is a two-variable function determined by the equation  $z - y - x + xe^{z-y-x} = 0$ . Find  $dz$ .

Solution:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy,$$

where

$$F'_x = -1 + e^{z-y-x} - xe^{z-y-x}, \quad F'_y = -1 - xe^{z-y-x}, \quad F'_z = 1 + xe^{z-y-x}.$$

Substituting into the formula:

$$\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = -\frac{F'_x}{F'_z} dx - \frac{F'_y}{F'_z} dy = \frac{1 + (x-1)e^{z-y-x}}{1 + xe^{z-y-x}} + 1.$$

**Example:** Given the total differential  $dz = 2x dx + \sin y dy$  for a function  $z = f(x, y)$  and  $f(1, 0) = 2$ , find  $f(x, y)$ .

Solution:

$$\because dz = 2x dx + \sin y dy, \quad \therefore \frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = \sin y.$$

Integrating the partial derivatives:

$$f(x, y) = x^2 + \varphi(y), \quad \frac{\partial(x^2 + \varphi(y))}{\partial y} = \varphi'(y) = \sin y.$$

Solving  $\varphi(y)$  yields:

$$\varphi(y) = -\cos y + C, \quad f(x, y) = x^2 - \cos y + C.$$

Substituting  $f(1, 0) = 2$ :

$$C = 2, \quad f(x, y) = x^2 - \cos y + 2.$$

## 3.4 Gradient and Directional Derivatives

### 3.4.1 Definition of Directional Derivative

For a bivariate function  $z = f(x, y)$  defined in a neighborhood around the point  $(x_0, y_0)$ , let  $\vec{u} = (\cos \alpha, \cos \beta)$  be a unit vector representing the direction. Then, if the limit

$$\lim_{\rho \rightarrow 0^+} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta) - f(x_0, y_0)}{\rho}$$

exists, it is called the directional derivative of the function  $z = f(x, y)$  at the point  $(x_0, y_0)$  along the direction  $\vec{u}$ , denoted as

$$D_{\vec{u}}f(x_0, y_0) = \lim_{t \rightarrow 0^+} \frac{f(x_0 + t \cos \alpha, y_0 + t \cos \beta) - f(x_0, y_0)}{t} = \left. \frac{\partial f(x, y)}{\partial \vec{u}} \right|_{(x_0, y_0)}.$$

**Note:** From a practical and general perspective, the definition of the directional derivative here uses  $t \rightarrow 0^+$ , meaning the directional derivative can take both positive and negative values. Although in some textbooks, the definition uses  $t \rightarrow 0$ , this generally applies to special cases. If a function is defined along a direction, and the directional derivatives exist and are equal when approaching from both directions, then the limit using  $t \rightarrow 0$  gives valid results. For general vectors and directions, the directional derivative can also be defined as:

$$D_{\vec{u}}f(X) = \frac{\partial f(X)}{\partial \vec{u}} = \lim_{t \rightarrow 0^+} \frac{f(X + t \cdot \vec{u}) - f(X)}{t}.$$

### 3.4.2 Geometric Interpretation of the Directional Derivatives

Let  $z = f(x, y)$  represent a surface  $S$ . The directional derivative  $D_{\vec{u}}f(x_0, y_0)$  represents the slope of the line of intersection of the plane parallel to the  $xOy$  plane in the direction of  $\vec{u}$  and perpendicular to the  $xOy$  plane with the surface  $S$ , at the point  $M(x_0, y_0, f(x_0, y_0))$ .

**Note:** Specifically,  $f_x(x_0, y_0)$  and  $f_y(x_0, y_0)$  are the directional derivatives of the function  $f(x, y)$  at the point  $(x_0, y_0)$  along the coordinate directions  $\vec{i} = (1, 0)$  and  $\vec{j} = (0, 1)$ , respectively.

### 3.4.3 Computation of Directional Derivatives

If the function  $f(x, y)$  is differentiable at point  $P(x_0, y_0)$ , then the directional derivative of the function along any direction vector  $\vec{u}$  at this point exists and can be computed as follows:

$$\begin{aligned}
 D_{\vec{u}}f(x_0, y_0) &= f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \cos \beta \\
 &= (f'_x(x_0, y_0), f'_y(x_0, y_0)) \cdot (\cos \alpha, \cos \beta) \\
 &= (f'_x(x_0, y_0), f'_y(x_0, y_0)) \cdot \vec{u} \\
 &= f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \sin \alpha
 \end{aligned}$$

where  $\cos \alpha, \cos \beta$  are the direction cosines of the unit vector  $\vec{u}$ . Since  $\alpha$  and  $\beta$  are complementary angles, the unit vector  $\vec{u}$  can also be written as  $(\cos \alpha, \sin \alpha)$ .

Thus, the directional derivative along  $\vec{u}$  is given by:

$$D_{\vec{u}}f(x_0, y_0) = f'_x \cos \alpha + f'_y \sin \alpha$$

### 3.4.4 Gradient and Directional Derivatives

The gradient of a two-variable function  $f(x, y)$  and a three-variable function  $f(x, y, z)$ , denoted by  $\nabla f$  or  $\text{grad} f$ , is defined as:

For two variables:

$$\nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

For three variables:

$$\nabla f(x, y, z) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

The directional derivative is the projection of the gradient in the direction of the unit vector  $\vec{u}$ :

$$D_{\vec{u}}f = \nabla f \cdot \vec{u}$$

1. The gradient direction is the direction of the fastest increase of the function value, while the opposite direction is the direction of the fastest decrease. Usually, the direction of the fastest increase is called the steepest ascent direction, and the direction of the fastest decrease is called the steepest descent direction.
2. At local maxima or minima of a function, the gradient is a zero vector.

3. The direction forming an acute angle with the gradient is the direction of increasing function values, while the direction forming an obtuse angle with the gradient is the direction of decreasing function values.
4. The gradient of a two-variable function and a three-variable function corresponds to the normal vector of the level curve and the level surface, respectively.

## 3.5 Geometric Applications of Multivariable Differentiations

### 3.5.1 Tangent Plane and Normal Line of a Surface

- **Surface in Space:**  $\Sigma : F(x, y, z) = 0$

The normal vector at point  $P_0(x_0, y_0, z_0)$  on the surface can be taken as:

$$\vec{n} = (F_x, F_y, F_z)_{P_0}$$

Based on the point-normal form of a plane, the equation of the tangent plane at point  $P_0$  on the surface is:

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

According to the point-normal form of a line, the equation of the normal line is:

$$\frac{x - x_0}{F_x(P_0)} = \frac{y - y_0}{F_y(P_0)} = \frac{z - z_0}{F_z(P_0)}$$

- **Surface in Space:**  $\Sigma : z = f(x, y)$

The surface equation can be rewritten in the form of a three-variable equation:

$$F(x, y, z) = \pm[f(x, y) - z] = 0$$

The normal vector at point  $P_0(x_0, y_0, z_0)$  on the surface can be taken as:

$$\vec{n} = \pm(f_x(x_0, y_0), f_y(x_0, y_0), -1)$$

The equation of the tangent plane at point  $P_0$  on the surface is:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = z - z_0$$

The equation of the normal line at point  $P_0$  is:

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}$$

### 3.5.2 Tangent Line and Normal Plane of a Space Curve

Given the parametric equations of the space curve:

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

The tangent direction vector at point  $(x(t_0), y(t_0), z(t_0))$  on the curve can be taken as:

$$\vec{T} = (x'(t_0), y'(t_0), z'(t_0))$$

Thus, the equation of the tangent line is:

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

The equation of the normal plane is:

$$x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) + z'(t_0)(z - z(t_0)) = 0$$

## 3.6 Extrema of Multivariable Functions

### 3.6.1 Concepts

**Definition:**

If there exists a neighborhood of  $(x_0, y_0)$  such that for any point  $(x, y)$  in this neighborhood, either  $f(x, y) \leq f(x_0, y_0)$  or  $f(x, y) \geq f(x_0, y_0)$  holds, then  $(x_0, y_0)$  is called a **generalized maximum/minimum point** of  $f(x, y)$ , and  $f(x_0, y_0)$  is called a **generalized maximum/minimum value** of  $f(x, y)$ .

If there exists a punctured neighborhood of  $(x_0, y_0)$  such that for any point  $(x, y)$  in this neighborhood, either  $f(x, y) < f(x_0, y_0)$  or  $f(x, y) > f(x_0, y_0)$  holds, then  $(x_0, y_0)$  is called a **true maximum/minimum point** of  $f(x, y)$ , and  $f(x_0, y_0)$  is called a **true maximum/minimum value** of  $f(x, y)$ .

Let  $(x_0, y_0)$  be a point in the domain of  $f(x, y)$ . If for any point  $(x, y)$  in the domain of  $f(x, y)$ , either  $f(x, y) \leq f(x_0, y_0)$  or  $f(x, y) \geq f(x_0, y_0)$  holds, then  $(x_0, y_0)$  is called a

**global maximum/minimum point** of  $f(x, y)$ , and  $f(x_0, y_0)$  is called a **global maximum/minimum value**.

Let  $(x_0, y_0)$  be a point in the domain of  $f(x, y)$ . If for any point  $(x, y)$  different from  $(x_0, y_0)$  in the domain of  $f(x, y)$ , either  $f(x, y) < f(x_0, y_0)$  or  $f(x, y) > f(x_0, y_0)$  holds, then  $(x_0, y_0)$  is called a **true global maximum/minimum point** of  $f(x, y)$ , and  $f(x_0, y_0)$  is called a **true global maximum/minimum value**.

A point where the first-order derivative of a function is zero is called a **stationary point** (also known as a critical point). For multivariable functions, a stationary point is a point where all first-order partial derivatives are zero.

A critical point is not necessarily an extremum: only differentiable extrema can be critical points. Similarly, critical points are not necessarily extrema: only critical points where the sign changes can become extrema.

**Theorem:** If  $f(x, y)$  achieves a local maximum or minimum at  $(x_0, y_0)$ , then  $f(x, y)$  achieves a local maximum or minimum at  $x = x_0$ , and  $f(x_0, y)$  achieves a local maximum or minimum at  $y = y_0$ .

### 3.6.2 Unconstrained Extrema

**Theorem:**

Necessary condition for extrema of a bivariate function: Let  $z = f(x, y)$  have first-order partial derivatives at  $(x_0, y_0)$ . If  $f(x, y)$  has an extremum at  $(x_0, y_0)$ , then  $f'_x(x_0, y_0) = 0$  and  $f'_y(x_0, y_0) = 0$ . This can be generalized to functions of three or more variables.

Sufficient condition for extrema of a bivariate function: If the second-order partial derivatives are given by

$$\begin{cases} f''_{xx}(x_0, y_0) = A \\ f''_{xy}(x_0, y_0) = B \\ f''_{yy}(x_0, y_0) = C \end{cases}$$

then the discriminant  $\Delta = B^2 - AC$  determines the type of extremum as follows:

$$\Delta = \begin{cases} < 0 & \Rightarrow \text{Extremum: } \begin{cases} A < 0 & \Rightarrow \text{Maximum} \\ A > 0 & \Rightarrow \text{Minimum} \end{cases} \\ > 0 & \Rightarrow \text{No extremum} \\ = 0 & \Rightarrow \text{Method fails, use definition} \end{cases}$$

This method only applies to bivariate functions.

### Explicit Functions

Explicit functions allow direct computation of  $f''_{xx}$ ,  $f''_{xy}$ , and  $f''_{yy}$ . First, set the first-order partial derivatives to zero to find candidate points. Then substitute these points into the second-order partial derivatives.

**Example:**

Find the extrema of the function  $f(x, y) = x^4 + y^4 - (x + y)^2$ .

Solution: Set  $f'_x = 4x^3 - 2(x + y) = 0$  and  $f'_y = 4y^3 - 2(x + y) = 0$ . Solving yields  $x = y = -1, 0, 1$ .

Compute the second-order partial derivatives:

$$f''_{xx} = 12x^2 - 2, \quad f''_{xy} = -2, \quad f''_{yy} = 12y^2 - 2$$

Substituting the points:

At  $P_1$ :  $A_1 = 10, B_1 = -2, C_1 = 10, \Delta = B_1^2 - A_1C_1 = -96 < 0, A_1 > 0$ , indicating a minimum.

Similarly,  $P_3$  is also a minimum. The minimum value is  $-2$ .

At  $P_2$ :  $A_2 = -2, B_2 = -2, C_2 = -2, \Delta_2 = B_2^2 - A_2C_2 = 0$ . The method fails.

Taking the path  $y = x$ , we have  $f(x, y) = f(x, x) = 2x^4 - 4x^2 = 2x^2(x + \sqrt{2})(x - \sqrt{2}) < 0$ .

Taking the path  $y = -x$ , we have  $f(x, y) = f(x, -x) = 2x^4 > 0$ , while  $f(0, 0) = 0$ .

Thus, along different paths, values greater and smaller than  $f(0, 0)$  exist, so  $(0, 0)$  is not an extremum.

### Implicit Functions

For implicit functions where  $f''_{xx}$ ,  $f''_{xy}$ , and  $f''_{yy}$  cannot be directly obtained, first set the first-order partial derivatives to zero to find candidate points. Then substitute these points into  $f'_x = 0$ ,  $f'_y = 0$ , and  $f'_z = 0$ .

### 3.6.3 Constrained Extrema and Lagrange Multipliers

To find the extrema of a target function  $u = f(x_1, x_2, \dots, x_n)$  subject to a set of constraint equations  $g(x_1, x_2, \dots, x_n) = 0$ :

1. Construct the Lagrangian function:

$$L(x_1, x_2, \dots, x_n, \lambda) = f(x_1, x_2, \dots, x_n) + \lambda g(x_1, x_2, \dots, x_n),$$

where  $g(x_1, x_2, \dots, x_n) = 0$  is the constraint.

2. Compute partial derivatives:

$$\begin{aligned}\frac{\partial L}{\partial x_i} &= 0, \quad i = 1, 2, \dots, n, \\ \frac{\partial L}{\partial \lambda} &= 0.\end{aligned}$$

3. Solve the system of equations formed by the partial derivatives and the constraint.

4. Determine the nature of the solution using the second-order conditions or by evaluating the objective function.

**Example:**

Find the minimum value of  $u = xyz$  subject to the constraint  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$ , where  $x > 0, y > 0, z > 0, a > 0$ .

Solution: Let

$$F(x, y, z, \lambda) = xyz + \lambda \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{a} \right)$$

Set the partial derivatives equal to zero:

$$F'_x = yz - \frac{\lambda}{x^2} = 0, \quad F'_y = xz - \frac{\lambda}{y^2} = 0, \quad F'_z = xy - \frac{\lambda}{z^2} = 0, \quad F'_\lambda = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{a} = 0$$

Solving yields  $x = y = z = 3a$ , so the minimum value is  $u_{\min} = 27a^3$ .

# Chapter 4

## Multiple Integral

### 4.1 Double Integrals

#### 4.1.1 Basic Concepts of Double Integrals

##### Geometric Background

The geometric background of a double integral is the volume of a curved-top cylinder. A definite integral uses the idea of limits to compute the area of a curved trapezoid in a two-dimensional plane. Similarly, a double integral  $\iint_D f(x, y) \, d\sigma$  computes volume.

The integrand  $f(x, y)$  represents the height of the infinitesimal cylinder above the point  $(x, y)$ , and  $d\sigma > 0$  is the base area. The product of the base area and height  $f(x, y)$  gives the volume of a small cylinder. Summing the volumes of all such cylinders over the domain  $D$  gives the total volume of the curved-top cylinder.

##### Properties

- **Area of a region:**  $\iint_D 1 \cdot d\sigma = \iint_D d\sigma = A$ , where  $A$  is the area of  $D$ .
- **Boundedness of integrable functions:** If  $f(x, y)$  is integrable over the bounded closed region  $D$ , then  $f(x, y)$  is bounded on  $D$ .
- **Linearity of integration:** For constants  $k_1, k_2$ ,

$$\iint_D [k_1 f(x, y) \pm k_2 g(x, y)] \, d\sigma = k_1 \iint_D f(x, y) \, d\sigma \pm k_2 \iint_D g(x, y) \, d\sigma.$$

- **Additivity of integration:** If  $f(x, y)$  is integrable over  $D$ , and  $D_1 \cup D_2 = D$ ,  $D_1 \cap D_2 = \emptyset$ , then

$$\iint_D f(x, y) \, d\sigma = \iint_{D_1} f(x, y) \, d\sigma + \iint_{D_2} f(x, y) \, d\sigma.$$

- **Sign preservation in integration:** If  $f(x, y)$  and  $g(x, y)$  are integrable over  $D$  and  $f(x, y) \leq g(x, y)$  over  $D$ , then

$$\iint_D f(x, y) \, d\sigma \leq \iint_D g(x, y) \, d\sigma.$$

In particular,

$$\left| \iint_D f(x, y) \, d\sigma \right| \leq \iint_D |f(x, y)| \, d\sigma.$$

- **Estimation theorem for double integrals:** Let  $M$  and  $m$  be the maximum and minimum values of  $f(x, y)$  over  $D$ , and  $A$  be the area of  $D$ . Then

$$mA \leq \iint_D f(x, y) \, d\sigma \leq MA.$$

- **Mean value theorem for double integrals:** If  $f(x, y)$  is continuous on the bounded closed region  $D$ , then there exists a point  $(\xi, \eta) \in D$  such that

$$\iint_D f(x, y) \, d\sigma = f(\xi, \eta)A.$$

**Example:**

Evaluate the order among  $I_1 = \iint_D \cos \sqrt{x^2 + y^2} \, d\sigma$ ,  $I_2 = \iint_D \cos(x^2 + y^2) \, d\sigma$ , and  $I_3 = \iint_D \cos(x^2 + y^2)^2 \, d\sigma$ , where  $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ .

A.  $I_3 > I_2 > I_1$

B.  $I_1 > I_2 > I_3$

C.  $I_2 > I_1 > I_3$

D.  $I_3 > I_1 > I_2$

Solution: Let  $x^2 + y^2 = t$ , then  $0 < t \leq 1$ . Thus,  $1 \geq \sqrt{t} \geq t \geq t^2 \geq 0$ . Since  $\cos x$  is monotonically decreasing, the order is  $I_3 > I_2 > I_1$ . Hence, the answer is **A**.

## Symmetry

**Ordinary Symmetry:** Let  $D$  be symmetric about the  $y$ -axis. Divide  $D$  into two symmetric parts  $D_1$  and  $D_2$ . For  $\iint_D f(x, y) d\sigma$ , we have:

$$I = \begin{cases} 2 \iint_{D_1} f(x, y) d\sigma, & f(x, y) = f(-x, y), \\ 0, & f(x, y) = -f(-x, y). \end{cases}$$

Similar results hold for  $x$ -axis symmetry.

**Rotational Symmetry:** If swapping  $x$  and  $y$  does not change  $D$ , or  $D$  is symmetric about  $y = x$ , then

$$\iint_D f(x, y) dx dy = \iint_D f(y, x) dy dx.$$

This is analogous to the variable invariance of single integrals:  $\int_a^b f(x) dx = \int_a^b f(y) dy$ .

**Example:**

Let  $D = \{(x, y) \mid x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$ , and  $f(x)$  be a continuous positive function on  $D$ . Compute

$$I = \iint_D \frac{a\sqrt{f(x)} + b\sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}} d\sigma.$$

Solution: The function is abstract, so direct computation is not possible. Since  $D$  is symmetric about  $y = x$ , we apply rotational symmetry:

$$I = \iint_D \frac{a\sqrt{f(x)} + b\sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}} d\sigma = \iint_D \frac{a\sqrt{f(y)} + b\sqrt{f(x)}}{\sqrt{f(y)} + \sqrt{f(x)}} d\sigma.$$

Thus:

$$2I = \iint_D \left( \frac{a\sqrt{f(x)} + b\sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}} + \frac{a\sqrt{f(y)} + b\sqrt{f(x)}}{\sqrt{f(y)} + \sqrt{f(x)}} \right) d\sigma = \iint_D (a + b) d\sigma = (a + b) \frac{\pi}{4}.$$

Hence,

$$I = \frac{a + b}{8} \pi.$$

### 4.1.2 Rectangular Coordinate System Calculation

When calculating double integrals, the limits of integration are determined by the region, and the integral is split into iterated integrals. This requires transforming the integral over a region into separate integrals with respect to  $x$  and  $y$ . The key is to determine the upper and lower bounds.

#### X-Type Region

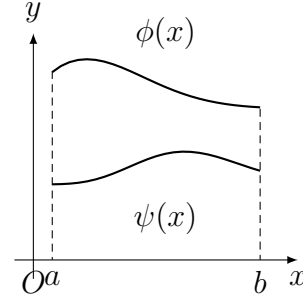
The region is defined as:

$$\sigma = \{(x, y) \mid a \leq x \leq b, \psi(x) \leq y \leq \phi(x)\}.$$

This is also known as an "up-down" region.

The double integral is expressed as:

$$\iint_D f(x, y) d\sigma = \int_a^b dx \int_{\psi(x)}^{\phi(x)} f(x, y) dy.$$



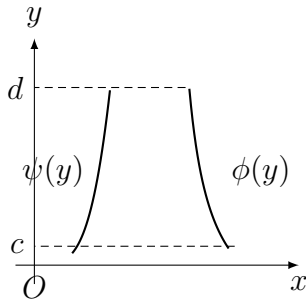
The double integral in an  $X$ -type region calculates the volume of a loaf-shaped object, as shown. To calculate the volume, consider the cross-sectional area. The horizontal cross-section at a given  $x$  has width  $\phi(x) - \psi(x)$  and height  $f(x, y)$ . The cross-sectional area is:

$$S(x) = \int_{\psi(x)}^{\phi(x)} f(x, y) dy.$$

By integrating the cross-sectional areas along the  $x$ -axis, the volume is:

$$V = \int_a^b S(x) dx.$$

#### Y-Type Region



The region is defined as:

$$\sigma = \{(x, y) \mid c \leq y \leq d, \psi(y) \leq x \leq \phi(y)\}.$$

This is also known as a "left-right" region.

The double integral is expressed as:

$$\iint_D f(x, y) d\sigma = \int_c^d dy \int_{\psi(y)}^{\phi(y)} f(x, y) dx.$$

### Choosing Region Type

If the boundaries are two curves in the  $y$ -direction, it is an  $X$ -type region. If the boundaries are two curves in the  $x$ -direction, it is a  $Y$ -type region.

If the same direction involves two different expressions, divide the region  $D$  into parts and integrate piecewise along the other direction.

### 4.1.3 Polar Coordinate System Calculation

Depending on the relationship between the integration region and the pole, the computation of double integrals can be divided into three cases:

The interval is divided based on the angle  $\theta$ , and from the pole, the region is divided along  $dr$ , resulting in shapes similar to small rectangles. One side of the shape is the radial increment  $dr$ , and the other side is an arc, which equals the radius multiplied by the angular increment  $r d\theta$ . Therefore,  $d\sigma = r dr d\theta$ .

In most cases, integrate with respect to  $r$  first, then integrate with respect to  $\theta$ .

The ray starts touching region  $D$  at  $\theta = \alpha$ , and it leaves the region at  $\theta = \beta$ . The ray moves between these two angles, denoted as  $\theta = \theta$ . The ray  $\theta = \alpha$  and  $\theta = \beta$  intersect the region  $D$  at two points. The boundary of  $D$  closer to the pole is called the inner curve, and the boundary farther from the pole is called the outer curve. The ray  $\theta = \theta$  intersects the inner curve at  $r = r_1(\theta)$  and the outer curve at  $r = r_2(\theta)$ .

1. When the pole  $O$  is outside region  $D$ :

$$\iint_D f(x, y) d\sigma = \int_{\alpha}^{\beta} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr.$$

2. When the pole  $O$  is on the boundary of region  $D$ :

$$\iint_D f(x, y) d\sigma = \int_{\alpha}^{\beta} d\theta \int_0^{r(\theta)} f(r \cos \theta, r \sin \theta) r dr.$$

3. When the pole  $O$  is inside region  $D$ :

$$\iint_D f(x, y) d\sigma = \int_0^{2\pi} d\theta \int_0^{r(\theta)} f(r \cos \theta, r \sin \theta) r dr.$$

### Conversion Between Polar and Rectangular Coordinates

When given a double integral, consider the following:

1. Check whether the integrand has the form  $f(x^2 + y^2)$ ,  $f\left(\frac{y}{x}\right)$ , or  $f\left(\frac{x}{y}\right)$ .
2. Check whether the integration region is a circle or part of a circle.
3. If either condition is satisfied, prefer using polar coordinates. Otherwise, consider using rectangular coordinates.

For converting from polar to rectangular coordinates, use:  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

**Example:**

Given the region  $D = \{(x, y) \mid x^2 + y^2 \leq R^2\}$ , compute  $\iint_D \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx dy$ .

Solution: Swap the integration variables:

$$I = \iint_D \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx dy = \iint_D \left(\frac{y^2}{a^2} + \frac{x^2}{b^2}\right) dx dy.$$

Thus:

$$2I = \left(\frac{1}{a^2} + \frac{1}{b^2}\right) \iint_D (x^2 + y^2) dx dy.$$

Therefore:

$$I = \frac{1}{2} \left(\frac{1}{a^2} + \frac{1}{b^2}\right) \iint_D (x^2 + y^2) d\sigma.$$

Using formula three and converting to polar coordinates:

$$I = \frac{1}{2} \left(\frac{1}{a^2} + \frac{1}{b^2}\right) \int_0^{2\pi} d\theta \int_0^R r^2 r dr.$$

Thus:

$$I = \left(\frac{1}{a^2} + \frac{1}{b^2}\right) \frac{\pi R^4}{4}.$$

**Example:** Compute  $I = \int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} \frac{x+y}{x^2+y^2} dy$ .

Solution: The upper bound  $\sqrt{1-x^2}$  and the lower bound  $1-x$  enclose a region  $D$  in the first quadrant, which is a circle minus a triangle.

When converting to polar coordinates, for  $\theta \in \left(0, \frac{\pi}{2}\right)$ ,  $r$  ranges from  $(1 - x, \sqrt{1 - x^2})$ . The lower bound  $x + y = 1$  becomes:

$$r \cos \theta + r \sin \theta = 1 \implies r = \frac{1}{\cos \theta + \sin \theta},$$

and the upper bound is a circle, so it is 1.

Thus:

$$I = \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\cos \theta + \sin \theta}}^1 (\cos \theta + \sin \theta) dr = \int_0^{\frac{\pi}{2}} (\cos \theta + \sin \theta - 1) d\theta = 2 - \frac{\pi}{2}.$$

#### 4.1.4 Calculation Techniques and Applications

##### Order of Integration

The order of integration refers to the selection of the region type, with the goal of simplifying the computation and making the integrand easier to evaluate.

In some cases, the integrand cannot be integrated in its current order, necessitating a change in the integration order.

There exist many functions with antiderivatives that cannot be expressed in elementary functions, such as  $\frac{\sin x}{x}$ ,  $\frac{\cos x}{x}$ ,  $\frac{\tan x}{x}$ ,  $\frac{e^x}{x}$ ,  $\sin x^2$ ,  $\cos x^2$ ,  $\tan x^2$ ,  $e^{ax^2+bx+c}$ ,  $\frac{1}{\ln x}$ , and others.

**Example:** Compute  $\int_1^2 dx \int_{\sqrt{x}}^x \sin\left(\frac{\pi x}{2y}\right) dy + \int_2^4 dx \int_{\sqrt{x}}^2 \sin\left(\frac{\pi x}{2y}\right) dy$ .

Solution: First, observe that the integrand is the same in both terms, but the regions of integration differ. This indicates a complex integration region.

The integrand is  $\sin\left(\frac{\pi x}{2y}\right)$ . Direct integration with respect to  $y$  is analogous to evaluating  $\int \sin\left(\frac{1}{x}\right) dx$ , which cannot be solved. Therefore, the order of integration must be changed to integrate with respect to  $x$  first.

Based on the integration bounds, the region is defined by  $\sqrt{x}$ ,  $x$ , and 2, forming a triangular-like region  $d\sigma$ .

$$I = \iint_D \sin\left(\frac{\pi x}{2y}\right) d\sigma = \int_1^2 dy \int_y^{y^2} \sin\left(\frac{\pi x}{2y}\right) dx = \int_1^2 \frac{2y}{\pi} \left(-\cos \frac{\pi y}{2} + \cos \frac{\pi}{2}\right) dy = \frac{4}{\pi^3}(2+\pi).$$

### Using Double Integrals to Handle Single Variable Integrals

Double integrals can be used when dealing with single-variable integrals involving intermediate variables.

**Example:** Let  $f(x) = \int_x^1 \sin(\pi u^2) du$ . Compute  $\int_0^1 f(x) dx$ .

Solution:

$$\int_0^1 f(x) dx = \int_0^1 dx \int_x^1 \sin(\pi u^2) du.$$

Since  $\sin(\pi u^2)$  cannot be integrated with respect to  $x$ , switch the order of integration. The region  $d\sigma$  is a triangle defined by  $x = 0$ ,  $x = 1$ , and  $u = x$ .

$$\begin{aligned} \int_0^1 dy \int_0^u \sin(\pi u^2) dx &= \int_0^1 \sin(\pi u^2) u du = \frac{1}{2\pi} \int_0^1 \sin(\pi u^2) d(\pi u^2). \\ &= -\frac{1}{2\pi} \cos \pi u^2 \Big|_0^1 = -\frac{1}{2\pi}(-1 - 1) = \frac{1}{\pi}. \end{aligned}$$

**Example:** Compute  $\int_0^{+\infty} e^{-x^2} dx$ .

Solution: Using the property of integration values being independent of variables:

$$I^2 = \left( \int_0^{+\infty} e^{-x^2} dx \right)^2 = \int_0^{+\infty} e^{-x^2} dx \cdot \int_0^{+\infty} e^{-y^2} dy = \int_0^{+\infty} \int_0^{+\infty} e^{-(x^2+y^2)} dx dy.$$

The region  $d\sigma$  is the first quadrant and can be treated as a generalized circle with infinite radius. Converting to polar coordinates:

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{+\infty} e^{-r^2} r dr = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta = \frac{\pi}{2}.$$

Thus:

$$I = \frac{\sqrt{\pi}}{2}.$$

### Centroid Formula

For rectangular coordinates and parametric equations:

$$\bar{x} = \frac{\iint_D x d\sigma}{\iint_D d\sigma} = \frac{\int x(t)y(t)x'(t) dt}{\int y(t)x'(t) dt}, \quad \bar{y} = \frac{\iint_D y d\sigma}{\iint_D d\sigma} = \frac{\int y(t)x'(t) dt}{\int y(t)x'(t) dt}.$$

## 4.2 Triple Integrals

### 4.2.1 Basic Concepts of Triple Integrals

The integrand  $f(x, y, z)$  of a triple integral is defined over a three-dimensional region  $\Omega$  and represents the volume of a four-dimensional geometric figure, which is highly abstract.

To provide a more tangible interpretation, consider a non-uniform object with volume density  $f(x, y, z)$ . The triple integral represents the total mass of the object, assuming  $f(x, y, z)$  is the density at each point.

**Definition:** Let the three-variable function  $f(x, y, z)$  be defined over a bounded and closed region  $\Omega$ . Divide the region  $\Omega$  into  $n$  subregions  $\Delta v_i$  ( $i = 1, 2, 3, \dots, n$ ), where  $\Delta v_i$  represents the volume of the  $i$ -th subregion. Select a point  $(\alpha_i, \beta_i, \gamma_i)$  in each  $\Delta v_i$  and form the sum  $\sum_{i=1}^n f(\alpha_i, \beta_i, \gamma_i) \Delta v_i$ . If the limit of this sum exists as the maximum diameter  $\lambda$  of all subregions approaches zero, this limit is defined as the triple integral of  $f(x, y, z)$  over  $\Omega$ . It is denoted as:

$$\iiint_{\Omega} f(x, y, z) dv = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta v_i,$$

where  $dv$  is called the volume element.

In this definition:

- $\iiint$  is the triple integral symbol,
- $f(x, y, z)$  is the integrand,
- $f(x, y, z) dv$  is the integral expression,
- $dv$  is the volume element,
- $x, y$ , and  $z$  are the integration variables,
- $\Omega$  is the integration region,
- $\sum f(\alpha_i, \beta_i, \gamma_i) \Delta v_i$  is the Riemann sum.

## Properties

Assume  $\Omega$  is a bounded and closed region in space.

- **Volume of a spatial region:** The volume  $V$  of the region  $\Omega$  is given by:

$$\iiint_{\Omega} 1 \, dv = \iiint_{\Omega} dv = V.$$

- **Boundedness of integrable functions:** If  $f(x, y, z)$  is integrable over  $\Omega$ , then  $f(x, y, z)$  must be bounded over  $\Omega$ .

- **Linearity of integrals:** Let  $k_1, k_2$  be constants. Then:

$$\iiint_{\Omega} [k_1 f(x, y, z) \pm k_2 g(x, y, z)] \, dv = k_1 \iiint_{\Omega} f(x, y, z) \, dv \pm k_2 \iiint_{\Omega} g(x, y, z) \, dv.$$

- **Additivity of integrals:** If  $f(x, y, z)$  is integrable over  $\Omega$ , and  $\Omega_1 \cup \Omega_2 = \Omega$  with  $\Omega_1 \cap \Omega_2 = \emptyset$ , then:

$$\iiint_{\Omega} f(x, y, z) \, dv = \iiint_{\Omega_1} f(x, y, z) \, dv + \iiint_{\Omega_2} f(x, y, z) \, dv.$$

- **Preservation of inequality:** If  $f(x, y, z)$  and  $g(x, y, z)$  are integrable over  $\Omega$ , and  $f(x, y, z) \leq g(x, y, z)$  over  $\Omega$ , then:

$$\iiint_{\Omega} f(x, y, z) \, dv \leq \iiint_{\Omega} g(x, y, z) \, dv,$$

and:

$$\left| \iiint_{\Omega} f(x, y, z) \, dv \right| \leq \iiint_{\Omega} |f(x, y, z)| \, dv.$$

- **Estimation theorem:** Let  $M$  and  $m$  be the maximum and minimum values of  $f(x, y, z)$  over  $\Omega$ , and let  $V$  be the volume of  $\Omega$ . Then:

$$mV \leq \iiint_{\Omega} f(x, y, z) \, dv \leq MV.$$

- **Mean value theorem:** If  $f(x, y, z)$  is continuous over  $\Omega$  and  $V$  is the volume of  $\Omega$ , then there exists a point  $(\xi, \eta, \zeta)$  in  $\Omega$  such that:

$$\iiint_{\Omega} f(x, y, z) \, dv = f(\xi, \eta, \zeta)V.$$

## Symmetry

The analysis of symmetry is identical to that of double integrals.

### Ordinary Symmetry

Assume  $\Omega$  is symmetric about the  $yOz$ -plane. Then:

$$\iiint_{\Omega} f(x, y, z) \, dv = \begin{cases} 2 \iiint_{\Omega_1} f(x, y, z) \, dv, & f(x, y, z) = f(-x, y, z), \\ 0, & f(x, y, z) = -f(-x, y, z), \end{cases}$$

where  $\Omega_1$  is the portion of  $\Omega$  in front of the  $yOz$ -plane.

### Rotational Symmetry

If swapping  $x$  and  $y$  leaves  $\Omega$  unchanged, then:

$$\iiint_{\Omega} f(x, y, z) \, dv = \iiint_{\Omega} f(y, x, z) \, dv.$$

This is known as **rotational symmetry**. Other cases can be handled similarly.

When using rotational symmetry, adjust the variables in the integrand  $f(x)$  based on the problem requirements, particularly swapping variables that do not appear in  $f(x)$ .

## 4.2.2 Rectangular Coordinate System Calculation

In rectangular coordinates,  $dv = dx dy dz$ , and the differential volume element is a rectangular cuboid.

### Method 1

This method integrates  $z$  first, followed by  $x$  and  $y$ . It involves performing a definite integral first, followed by a double integral. This approach essentially slices the solid vertically and integrates the volume of each slice to obtain the total volume. Therefore, this method requires that the base of the solid be fixed.

**Applicable Scenario:** The region  $\Omega$  has a lower surface  $z = z_1(x, y)$  and an upper surface  $z = z_2(x, y)$ , with no side surfaces or cylindrical side surfaces.

Similar to double integrals: perform integration by determining the limits, drawing lines within the limits, and first writing the lower limit followed by the upper limit.

$$\iiint_{\Omega} f(x, y, z) \, dv = \iint_{D_{xy}} d\sigma \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) \, dz.$$

**Example:**

Compute the triple integral  $I = \iiint_{\Omega} \frac{dx dy dz}{(1 + x + y + z)^3}$ , where  $\Omega$  is the tetrahedron bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ , and  $x + y + z = 1$ .

Solution: The given region is a tetrahedron. The base  $D_{xy}$  is an isosceles right triangle with sides of length 1, and the upper surface is the plane  $z = 1 - x - y$ . The region has two vertical side surfaces.

First, integrate with respect to  $z$ :

$$\begin{aligned} I &= \iint_{D_{xy}} d\sigma \int_0^{1-x-y} \frac{1}{(1 + x + y + z)^3} \, dz. \\ &= \iint_{D_{xy}} d\sigma \left( -\frac{1}{2} \frac{1}{(1 + x + y + z)^2} \Big|_{z=0}^{z=1-x-y} \right) = \iint_{D_{xy}} \frac{1}{2} \left( \frac{1}{(1 + x + y)^2} - \frac{1}{4} \right) d\sigma. \\ &= \frac{1}{2} \int_0^1 dx \int_0^{1-x} \left( \frac{1}{(1 + x + y)^2} - \frac{1}{4} \right) dy = \frac{1}{2} \int_0^1 \left( -\frac{1}{4}x + \frac{1}{1+x} - \frac{1}{4} \right) dx. \\ &= \left( -\frac{1}{8}x^2 + \ln(1+x) - \frac{1}{4}x \right) \Big|_0^1 = \frac{1}{2} \left( \ln 2 - \frac{5}{8} \right). \end{aligned}$$

## Method 2

This method integrates  $x$  and  $y$  first, followed by  $z$ . It involves performing a double integral first, followed by a definite integral. This approach slices the solid horizontally into cylindrical volumes, and integrating over all slices gives the total volume.

**Applicable Scenario:** The region  $\Omega$  is a solid of revolution, where the top and bottom surfaces are planes, and the middle surface is a curved surface with a rotational symmetry described by  $\Sigma : z = z(x, y)$ .

The process involves determining the limits of integration by slicing the region horizontally.

$$\iiint_{\Omega} f(x, y, z) \, dv = \int_a^b dz \iint_{D_z} f(x, y, z) \, d\sigma.$$

### 4.2.3 Cylindrical Coordinate System Calculation

If the double integral part  $\iint_{D_{xy}} d\sigma$  is suitable for polar coordinates (i.e., related to a circular region), use polar coordinates by letting  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then, the triple integral can be written as:

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega} f(r \cos \theta, r \sin \theta, z) r dr d\theta dz.$$

This is the method for computing triple integrals in cylindrical coordinates.

**Applicable Scenario:** The integrand contains terms involving  $x^2 + y^2$ , and the integration region is a circle or part of a circle.

In essence, it involves a single integral combined with a double integral in polar coordinates.

**Example:**

Compute the triple integral  $\iiint_{\Omega} (x^2 + y^2) dv$ , where  $\Omega$  is the region enclosed by the surface formed by rotating the curve  $\begin{cases} y^2 = 2z \\ x = 0 \end{cases}$  about the  $z$ -axis, and the plane  $z = 8$ .

**Solution:** Since the planar curve is rotated about the  $z$ -axis, we first determine the resulting surface of revolution.

Assume a point  $P_1(x_1, y_1, z_1)$  lies on the given curve, yielding the equations  $y_1^2 = 2z_1$  and  $x_1 = 0$ .

Select a point  $P_0(0, 0, 0)$  on the axis of rotation ( $z$ -axis). For any point  $P(x, y, z)$  on the resulting surface, the condition  $|P_0P_1| = |PP_0|$  holds, implying that:

$$x_1^2 + y_1^2 + z_1^2 = x^2 + y^2 + z^2.$$

Additionally, since the vector  $\overrightarrow{PP_1}$  is perpendicular to the axis of rotation, we have  $(x_1 - x, y_1 - y, z_1 - z) \perp (0, 0, 1)$ , which simplifies to:

$$z_1 - z = 0 \quad \Rightarrow \quad z_1 = z.$$

Substitute  $z_1 = z$  into the equation, yielding  $x_1^2 + y_1^2 = x^2 + y^2$ . Then, substituting  $y_1^2 = 2z$  and  $x_1 = 0$ , we obtain:  $2z = x^2 + y^2$ .

Thus, the surface of revolution is described by the equation:  $z = \frac{x^2 + y^2}{2}$ .

The region  $\Omega$  is bounded by this surface and the plane  $z = 8$ , forming a solid of revolution.

Since both the top and bottom of the solid are planes, and the side is a curved surface, we use the two-then-one method. The region  $D$  is given by:  $D : x^2 + y^2 \leq 2z$ .

The integral becomes:

$$I = \int_0^8 dz \iint_D (x^2 + y^2) d\sigma = \int_0^8 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{2z}} r^2 r dr = \frac{1024}{3} \pi.$$

#### 4.2.4 Spherical Coordinate System Calculation

If the integrand contains  $x^2 + y^2 + z^2$  or  $x^2 + y^2$ , and the region of integration is a sphere or part of a sphere, or a cone or part of a cone, spherical coordinates are used. The principle involves slicing  $\Omega$  with three types of surfaces:

1. First, use  $r = r_0$  to slice outward from the origin, where the radius is  $r_0$  with an increment of  $dr$ . Here,  $r_0 \in [0, +\infty)$ .
2. Next, use  $\phi = \phi_0$  to slice using cones with half-apex angle  $\phi_0$  centered around the  $z$ -axis, with an increment of  $d\phi$ . Here,  $\phi_0 \in [0, \pi]$ .
3. Finally, use  $\theta = \theta_0$  to slice using half-planes about the  $z$ -axis, with an angle  $\theta_0$  relative to the  $xOz$  plane, and an increment of  $d\theta$ . Here,  $\theta \in [0, 2\pi]$ .

In polar coordinates, the arc length is given by multiplying the radius by the angle. Thus, one side of the differential element determined by  $\phi$  is  $r d\phi$ . For the side determined by  $\theta$ , first use the Pythagorean theorem to get the arc length  $r \sin \phi$ , then multiply by  $d\theta$  to obtain the side length  $r \sin \phi d\theta$ . Finally, multiplying by  $dr$ , the differential volume element is:

$$r d\phi \cdot r \sin \phi d\theta \cdot dr = r^2 \sin \phi d\theta d\phi dr.$$

For  $x$ ,  $y$ , and  $z$ , form a right triangle with hypotenuse  $r$  and half-apex angle  $\phi$ . The perpendicular side along the  $z$ -axis is  $z = r \cos \phi$ . Since  $x^2 + y^2 + z^2 = r^2$ , we have  $x^2 + y^2 = r^2 - z^2 = r^2 \sin^2 \phi$ . Given that the angle in the  $xOy$  plane is  $\theta$ , we obtain:

$$x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta.$$

### Computation Method

$$\iiint_{\Omega} f(x, y, z) \, dx dy dz = \iiint_{\Omega} f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) r^2 \sin \phi \, d\theta d\phi dr.$$

Let  $x = r \cos \theta \sin \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \phi$ , and determine the angles  $\phi$  and  $\theta$  relative to the  $xOy$  and  $xOz$  planes, respectively.

**Example:** Compute the triple integral  $\iiint_{\Omega} (x^2 + y^2) \, dv$ , where  $\Omega$  is the region bounded by the hemisphere  $x^2 + y^2 + z^2 = a^2$  ( $y \geq 0$ ) and the  $xOz$  plane.

**Solution:** Since the region is a right hemisphere, the angle  $\theta$  varies from 0 to  $\pi$  (from the positive  $x$ -axis to the negative  $x$ -axis),  $\phi$  varies from 0 to  $\pi$  (from the positive  $z$ -axis to the negative  $z$ -axis), and  $r$  varies from 0 to  $a$ . The function  $f(x) = x^2 + y^2 = r^2 \sin^2 \phi$ .

$$\therefore I = \int_0^{\pi} d\theta \int_0^{\pi} d\phi \int_0^a (r^2 \sin^2 \phi) r^2 \sin \phi \, dr = \frac{\pi^4 a^5}{5}.$$

## 4.2.5 Symmetry of Triple Integrals

### Ordinary Symmetry

**Example:**

Compute the integral  $\iiint_{\Omega} e^{|z|} \, dv$ , where  $\Omega : x^2 + y^2 + z^2 \leq 1$ .

**Solution:** Since  $\Omega$  is a sphere of radius 1 centered at the origin, we can use ordinary symmetry:

$$f(x, y, z) \, dv = e^{|z|} \, dv = dm = f(x, y, -z) \, dv = e^{|z|} \, dv = dm,$$

which implies that the integral over the upper and lower hemispheres is the same.

$$\therefore \iiint_{\Omega} e^{|z|} \, dv = 2 \iiint_{\Omega_1} e^z \, dv,$$

where  $\Omega_1$  is the upper hemisphere, i.e.,  $\Omega_1 : x^2 + y^2 + z^2 \leq 1, z \geq 0$ .

Since  $f(x) = e^z$  depends only on  $z$ , it is easier to use the two-then-one method.

The cross-section is a circle, and substituting  $z = z$  into the equation of the sphere gives the area:

$$D : x^2 + y^2 \leq 1 - z^2,$$

so the radius squared is  $r^2 = 1 - z^2$ , and the area is  $\pi r^2 = \pi(1 - z^2)$ .

$$= 2 \int_0^1 dz \iint_D e^z \, d\sigma = 2 \int_0^1 e^z \cdot \pi(1 - z^2) \, dz = 2\pi.$$

**Rotational Symmetry****Example:**

Let  $\Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ . Compute  $\iiint_{\Omega} z^2 \, dx \, dy \, dz$ .

**Solution:** By rotational symmetry, we have:

$$\because \iiint_{\Omega} x^2 \, dx \, dy \, dz = \iiint_{\Omega} y^2 \, dx \, dy \, dz = \iiint_{\Omega} z^2 \, dx \, dy \, dz.$$

$$\therefore I = \frac{1}{3} \iiint_{\Omega} (x^2 + y^2 + z^2) \, dx \, dy \, dz.$$

# Chapter 5

## Line and Surface Integrals

### 5.1 Scalar Line Integrals

#### 5.1.1 Concept of Scalar Line Integrals

Scalar line integrals are an extension of definite integrals, specifically integrals along an arc length. They can be used to compute the mass of irregularly shaped wires with non-uniform density.

#### Geometric Properties

Let  $L$  be a curve segment on the  $xOy$  plane, and  $f(x, y)$  be a bounded function on  $L$ . Divide  $L$  into multiple segments  $\Delta S_1, \Delta S_2, \dots, \Delta S_n$ . Suppose a point  $(\xi_i, \eta_i) \in \Delta S_i$  is selected on each segment. The mass of the segment can be approximated as  $\Delta m_i \approx \rho(\xi_i, \eta_i) \Delta S_i$ .

Therefore, the total mass  $m \approx \sum_{i=1}^n \rho(\xi_i, \eta_i) \Delta S_i$ . If  $\lambda = \max\{\Delta S_1, \dots, \Delta S_n\}$  and the limit

$m = \lim_{\delta \rightarrow 0} \sum_{i=1}^n \rho(\xi_i, \eta_i) \Delta S_i$  exists, this limit is defined as the scalar line integral of  $f(x, y)$  along  $L$ , denoted by  $\int_L f(x, y) ds$ .

#### Definition

The scalar line integral is denoted as  $\int_L f(x, y) ds$ .

**Properties**

- $\int_L 1 \, ds = l.$
- $\int_L f(x, y) \, ds = \int_{L_1} f(x, y) \, ds + \int_{L_2} f(x, y) \, ds.$
- $\int_L (k_1 f(x, y)) \, ds \pm \int_L (k_2 f(x, y)) \, ds = k_1 \int_L f(x, y) \, ds \pm k_2 \int_L f(x, y) \, ds.$

**Symmetry**

- If  $L$  is symmetric about the  $y$ -axis, with the right part denoted as  $L_1$ , and  $f(-x, y) = -f(x, y)$ , then  $\int_L f(x, y) \, ds = 0$ . If  $f(-x, y) = f(x, y)$ , then  $\int_L f(x, y) \, ds = 2 \int_{L_1} f(x, y) \, ds.$
- If  $L$  is symmetric about the  $x$ -axis, with the upper part denoted as  $L_1$ , and  $f(x, -y) = -f(x, y)$ , then  $\int_L f(x, y) \, ds = 0$ . If  $f(x, -y) = f(x, y)$ , then  $\int_L f(x, y) \, ds = 2 \int_{L_1} f(x, y) \, ds.$
- If  $L$  is symmetric about the line  $y = x$ , then  $\int_L f(x, y) \, ds = \int_L f(y, x) \, ds.$

**5.1.2 Computation of Scalar Line Integrals**

The computation involves converting the line integral into a definite integral. Project the curve onto an axis, substitute the parametric equations, and transform  $ds$  accordingly.

- If  $L : y = g(x) \ (a \leq x \leq b)$ , then  $\int_L f(x, y) \, ds = \int_a^b f[x, g(x)] \sqrt{1 + g'^2(x)} \, dx.$
- If  $L : \begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \ (\alpha \leq t \leq \beta)$ , then  $\int_L f(x, y) \, ds = \int_\alpha^\beta f[\phi(t), \psi(t)] \sqrt{\phi'^2(t) + \psi'^2(t)} \, dt.$

**Example:**

Compute  $\oint_\Gamma |y| \, ds$ , where  $\Gamma$  is the curve formed by the intersection of the sphere  $x^2 + y^2 + z^2 = 2$  and the plane  $x = y$ .

Solution: By ordinary symmetry, for  $|y|$ , the function values in other quadrants are equal to those in the first quadrant. Let  $\Gamma_1$  represent the curve in the first quadrant:

$$\therefore \oint_\Gamma |y| \, ds = 4 \oint_{\Gamma_1} y \, ds.$$

Using the parametric equations of the intersection, we have  $x = \cos t$ ,  $y = \cos t$ ,  $z = \sqrt{2} \sin t$ .

For  $\Gamma_1$ , the angle  $t \in \left(0, \frac{\pi}{2}\right)$ , and  $y = \cos t$ .

Since  $ds = \sqrt{x_t'^2 + y_t'^2 + z_t'^2}$ , we have  $ds = \sqrt{2} dt$ .

$$= 4 \int_0^{\frac{\pi}{2}} \cos t \sqrt{2} dt = 4\sqrt{2}.$$

## 5.2 Vector Line Integrals

Vector line integrals involve integrating along a curve in terms of coordinates. Before introducing vector line integrals, the concept of a field is explained.

### 5.2.1 Vector Fields

**Definition:** A vector field is a mapping rule defined over a spatial region  $\Omega$ .

A scalar field assigns a scalar value without direction, while a vector field assigns both magnitude and direction.

### 5.2.2 Geometric Properties

In an idealized situation where a force  $\vec{F}$  acts uniformly along a straight path  $\overrightarrow{AB}$ , the work done is given by  $\vec{F} \cdot \overrightarrow{AB}$ , where  $\vec{F}$  is the force vector and  $\overrightarrow{AB}$  is the displacement vector.

In a non-ideal situation where the force varies along a curve  $L$ , the work done cannot be directly computed using the above formula.

By dividing the curve  $L$  into infinitesimal segments  $\forall \vec{ds} \in L$ , we have  $\vec{ds} = \{dx, dy\}$ . Let the varying force  $\vec{F}(x, y) = \{P(x, y), Q(x, y)\}$ . The infinitesimal work done is  $d\omega = \vec{F} \cdot \vec{ds} = P(x, y)dx + Q(x, y)dy$ . Therefore, the total work along the curve is given by:

$$\omega = \int_L d\omega = \int_L P(x, y)dx + Q(x, y)dy.$$

In three dimensions, divide the curve  $L$  into infinitesimal segments  $\forall \vec{ds} \in L$ , where  $\vec{ds} = \{dx, dy, dz\}$ . Let the varying force  $\vec{F}(x, y, z) = \{P(x, y, z), Q(x, y, z), R(x, y, z)\}$ . The infinitesimal work done is:

$$d\omega = \vec{F} \cdot \vec{ds} = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz,$$

and the total work along the curve is:

$$\omega = \int_L d\omega = \int_L P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz.$$

### 5.2.3 Definition of Vector Line Integrals

In two dimensions, the line integral along a curve is:

$$\int_L P(x, y)dx + Q(x, y)dy,$$

where  $\int_L P(x, y) dx$  represents the integral of  $P(x, y)$  with respect to  $x$  along the oriented curve  $L$ , and  $\int_L Q(x, y) dy$  represents the integral of  $Q(x, y)$  with respect to  $y$  along  $L$ .

In three dimensions, the line integral along a curve is:

$$\int_L P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz,$$

following a similar definition as in two dimensions.

### 5.2.4 Properties of Vector Line Integrals

- $\int_{L^-} P(x, y) dx + Q(x, y) dy = - \int_L P(x, y) dx + Q(x, y) dy.$
- $\int_L P(x, y) dx + Q(x, y) dy = \int_L (P(x, y) \cos \alpha + Q(x, y) \cos \beta) ds,$  where the direction cosines are used to transform a second-kind line integral into a first-kind line integral.

### 5.2.5 Computation of Vector Line Integrals by Definite Integrals

For non-closed curves, use the definite integral method. For closed curves or parts of closed curves, use the double integral method (will be discussed in the next section).

Since the value of a vector line integral can be positive or negative, only the starting point  $x = a$  and the endpoint  $x = b$  need to be considered.

- If  $L : y = g(x)$  (where  $x$  starts at  $a$  and ends at  $b$ ), then:

$$\int_L P(x, y) dx + Q(x, y) dy = \int_a^b P[x, g(x)] dx + Q[x, g(x)]g'(x) dx.$$

- If  $L : \begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases}$  (where  $t$  starts at  $\alpha$  and ends at  $\beta$ ), then:

$$\int_L P(x, y) dx + Q(x, y) dy = \int_{\alpha}^{\beta} P[\phi(t), \psi(t)]\phi'(t) dt + Q[\phi(t), \psi(t)]\psi'(t)dt.$$

## 5.3 Green's Theorem

### 5.3.1 Background of Green's Theorem

A region  $G$  in the complex plane is called simply connected if any simple closed curve within  $G$  always has its interior entirely contained in  $G$ . A region that is not simply connected is called multiply connected. Intuitively, a complete convex shape is simply connected, while a shape with holes or concave parts is multiply connected.

For simply connected regions, the positive orientation of the boundary is counterclockwise. For multiply connected regions, the positive orientation of the outer boundary is counterclockwise, while the inner boundary is clockwise.

In definite integrals, Newton-Leibniz's formula is used to transform  $\int_a^b f(x) dx = F(b) - F(a)$ , which converts the integral value over the range  $[a, b]$  into the difference of the function values at the boundaries. In double integrals, the integral region is a connected domain, and the area integral represents the double integral, while the perimeter integral represents the line integral. Thus, a method that connects the domain relation with the boundary relation is sought—specifically, a method that links double integrals and line integrals.

Green's theorem provides a tool to convert line integrals into double integrals.

### 5.3.2 Conditions for Applying Green's Theorem

- $D$  is a connected region, and  $L$  is the positively oriented boundary of  $D$ . (If  $D$  is not a complete region, add a line segment; if  $L$  is negatively oriented, add a negative sign.)
- $P(x, y)$  and  $Q(x, y)$  are continuously differentiable in  $D$ .

$$\oint_L P(x, y) dx + Q(x, y) dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d\sigma.$$

### 5.3.3 Path-Independent Integrals

For vector line integrals  $\int_L P(x, y) dx + Q(x, y) dy$ , the value of the integral may or may not depend on the path  $L$ .

If  $D$  is simply connected and  $P(x, y)$ ,  $Q(x, y)$  are continuously differentiable in  $D$ , the following statements are equivalent:

1.  $\int_L P(x, y) dx + Q(x, y) dy$  is path-independent.
2.  $\forall C \in D$  (where  $C$  is a closed region),  $\oint_C P(x, y) dx + Q(x, y) dy = 0$ .
3. Cauchy-Riemann condition:  $\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y}$  (most suitable condition).
4.  $\exists u(x, y)$  such that  $du = P(x, y) dx + Q(x, y) dy$ , i.e.,  $\frac{\partial u}{\partial x} = P(x, y)$  and  $\frac{\partial u}{\partial y} = Q(x, y)$ .

These conditions can simplify calculations. If the Cauchy-Riemann condition is satisfied:

1. To compute the line integral:

$$\int_L P(x, y) dx + Q(x, y) dy = \int_{(x_0, y_0)}^{(x_1, y_1)} P(x, y) dx + Q(x, y) dy = \int_{x_0}^{x_1} P(x, y_0) dx + \int_{y_0}^{y_1} Q(x_1, y) dy,$$

performing horizontal integration first and then vertical integration.

2. If  $u$  can be found, compute the line integral:

$$\int_L P(x, y) dx + Q(x, y) dy = \int_{(x_0, y_0)}^{(x_1, y_1)} du = u(x_1, y_1) - u(x_0, y_0).$$

3. To find the total differential corresponding to  $u$ :

$$u(x, y) = \int_{(x_0, y_0)}^{(x, y)} P(x, y) dx + Q(x, y) dy = \int_{x_0}^x P(x, y_0) dx + \int_{y_0}^y Q(x_1, y) dy,$$

performing horizontal integration first and then vertical integration.

**Example:**

Given that  $\psi(x)$  is differentiable and  $\psi(0) = 2$ , and that  $\int_L xy^2 dx + \psi(x)y dy$  is path-independent, find  $\psi(x)$  and compute  $I = \int_{(1,2)}^{(2,3)} xy^2 dx + \psi(x)y dy$ .

Solution: Given  $P = xy^2$  and  $Q = \psi(x)y$ . Since the line integral is path-independent,  $\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y}$ . Therefore,  $\psi'(x)y \equiv 2xy$ , which implies  $\psi'(x) = 2x$ , giving  $\psi(x) = x^2 + C$ . Since  $\psi(0) = 2$ ,  $C = 2$ , so  $\psi(x) = x^2 + 2$ .

Thus,

$$I = \int_{(1,2)}^{(2,3)} xy^2 dx + (x^2 + 2)y dy.$$

Method 1:

$$I = \int_1^2 x \cdot 2^2 dx + \int_2^3 (2^2 + 2)y dy = 2x^2 \Big|_1^2 + 3y^2 \Big|_2^3 = 21.$$

Method 2:

$$I = \int_{(1,2)}^{(2,3)} (xy^2 dx + x^2y dy) + 2y dy = \int_{(1,2)}^{(2,3)} d\left(\frac{1}{2}x^2y^2 + y^2\right) = \left(\frac{1}{2}x^2y^2 + y^2\right) \Big|_{(1,2)}^{(2,3)} = 21.$$

## 5.4 Surface Integrals of a Scalar-Valued Function

### 5.4.1 Concept of Surface Integral Over Area

#### Geometric Properties

If an infinitesimally thin object  $\Sigma$  exists in space, taking an area element  $\forall ds \in \Sigma$  and assuming uniform surface density, the mass of the element is given by  $dm = \rho(x, y, z)ds$ .

Therefore, the total mass is  $m = \iint_{\Sigma} dm = \iint_{\Sigma} \rho(x, y, z) ds$ .

#### Definition

$\iint_{\Sigma} f(x, y, z) ds$  represents the surface integral of  $f(x, y, z)$  over the surface  $\Sigma$ .

#### Properties

- $\iint_{\Sigma} 1 ds = \sigma$ .
- If  $\Sigma$  is symmetric about the  $xOy$  plane, and the upper part is  $\Sigma_1$ , then:
  - If  $f(x, y, -z) = -f(x, y, z)$ , then  $\iint_{\Sigma} f(x, y, z) ds = 0$ .
  - If  $f(x, y, -z) = f(x, y, z)$ , then  $\iint_{\Sigma} f(x, y, z) ds = 2 \iint_{\Sigma_1} f(x, y, z) ds$ .

### 5.4.2 Computation of Scalar Line Integrals

Use the same method in computation of scalar line integrals. Project the curve onto an axis, substitute the parametric equations, and transform  $ds$  accordingly.

1. Draw the surface  $\Sigma$ .
2. Check the symmetry of the shape. If it is symmetric, eliminate odd terms in  $x$ ,  $y$ , or  $z$ . For example, if the surface is symmetric about the  $xOy$  plane, eliminate terms involving odd powers of  $z$ . Similarly, eliminate odd powers of  $x$  for symmetry about the  $yOz$  plane, and odd powers of  $y$  for symmetry about the  $xOz$  plane.
3. Check whether constant terms can simplify the expression.
4. Project  $\Sigma$  onto the  $xOy$  plane, where  $z = \psi(x, y)$  and  $(x, y) \in D_{xy}$ .
5. Compute  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ , and use  $ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} d\sigma$ .
6. Substitute into the integral:

$$\iint_{\Sigma} f(x, y, z) ds = \iint_{D_{xy}} f[x, y, \psi(x, y)] \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} d\sigma.$$

**Example:**

Compute  $I = \iint_{\Sigma} (xy^2 + z) ds$ , where  $\Sigma : z = x^2 + y^2, z \leq 1$ .

Solution: The surface  $\Sigma$  is an upward-opening paraboloid. It is not symmetric about the  $z$ -axis but is symmetric about the  $x$ -axis and  $y$ -axis. Since  $I$  contains the term  $xy^2$  involving an odd power of  $x$ , we eliminate it. Since there are no odd powers of  $y$ , we proceed.

$$\therefore I = \iint_{\Sigma} z ds.$$

Projecting  $\Sigma$  onto the  $xOy$  plane yields a unit disk  $D_{xy} : x^2 + y^2 \leq 1$ .

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y, \quad ds = \sqrt{1 + 4x^2 + 4y^2} d\sigma.$$

$$I = \iint_{D_{xy}} (x^2 + y^2) \sqrt{1 + 4(x^2 + y^2)} d\sigma = \int_0^{2\pi} d\theta \int_0^1 r^3 \sqrt{1 + 4r^2} dr = \pi \int_0^1 r^2 \sqrt{1 + 4r^2} d(r^2).$$

Let  $r^2 = t$ , then:

$$I = \pi \int_0^1 t \sqrt{1+4t} dt.$$

Let  $\sqrt{1+4t} = u$ , then:

$$I = \pi \int_1^{\sqrt{5}} \frac{u^2-1}{4} u \frac{u}{2} du = \frac{\pi}{8} \int_1^{\sqrt{5}} (u^4 - u^2) du = \frac{\pi}{8} \left( \frac{u^5}{5} - \frac{u^3}{3} \right) \Big|_1^{\sqrt{5}} = \frac{10}{3} \sqrt{5} + \frac{2}{15}.$$

**Example:**

Given the surface  $\Sigma : |x| + |y| + |z| = 1$ , compute  $\oiint_{\Sigma} (x + |y|) ds$ .

Solution: The surface  $\Sigma$  is a regular octahedron. By symmetry,  $\oiint_{\Sigma} x ds = 0$ .

Let the first octant be  $\Sigma_1$ . Using symmetry:

$$\oiint_{\Sigma} |y| ds = 8 \iint_{\Sigma_1} y ds.$$

Since  $x + y + z = 1$  and  $ds = \sqrt{1 + z_x'^2 + z_y'^2} dx dy$ , interchanging  $x$  and  $y$  keeps  $xy$  unchanged.

By rotational symmetry:

$$\iint_{\Sigma_1} y ds = \iint_{\Sigma_1} x ds.$$

Since  $x + y + z = 1$ , interchanging  $x$  and  $z$  also keeps  $xz$  unchanged.

Therefore:

$$\iint_{\Sigma_1} y ds = \iint_{\Sigma_1} x ds = \iint_{\Sigma_1} z ds.$$

$$\therefore 8 \iint_{\Sigma_1} y ds = \frac{8}{3} \iint_{\Sigma_1} (x + y + z) ds = \frac{8}{3} \iint_{\Sigma_1} ds = \frac{8}{3} S_{\Sigma_1} = \frac{8}{3} \cdot \frac{\sqrt{3}}{2} = \frac{4}{3} \sqrt{3}.$$

## 5.5 Surface Integrals of a Vector Field

### 5.5.1 Flux of a Vector Field

Consider an incompressible vector field  $\vec{v} = \{P, Q, R\}$  and a surface  $\Sigma$  within the vector field, where fluid flows irregularly into or out of the surface. We aim to calculate the flux flowing into a specified side per unit time.

Take an oriented surface element  $\forall \vec{ds} \in \Sigma$ . Projecting  $\vec{ds}$  onto the  $yOz$  plane from the positive  $x$ -axis direction gives  $dydz$ .

When  $\cos \alpha \geq 0$ , i.e., the directional cosine is non-negative,  $dydz = ds_{yOz}$ , which equals the projected area. When  $\cos \alpha < 0$ , i.e., the directional cosine is negative,  $dydz = -ds_{yOz}$ , which equals the negative projected area. In general,  $dydz = ds \cdot \cos \alpha$ . The same principle applies for the other two directions.

Thus, projecting  $\vec{ds}$  onto all three axes yields:

$$\vec{ds} = \{dydz, dzdx, dxdy\}.$$

Therefore, the flux through  $\vec{ds}$  into a specified side per unit time is:

$$d\phi = \vec{v} \cdot \vec{ds} = P(x, y, z)dydz + Q(x, y, z)dzdx + R(x, y, z)dxdy.$$

The total flux is given by:

$$\iint_{\Sigma} d\phi = \iint_{\Sigma} P(x, y, z)dydz + Q(x, y, z)dzdx + R(x, y, z)dxdy.$$

### 5.5.2 Definition and Properties of Surface Integrals of a Vector Field

The expression  $\iint_{\Sigma} P(x, y, z)dydz + Q(x, y, z)dzdx + R(x, y, z)dxdy$  represents the surface integral over  $\Sigma$ . Specifically,  $\iint_{\Sigma} P(x, y, z)dydz$  represents the surface integral of  $P(x, y, z)$  over the oriented surface  $\Sigma$  with respect to the  $y$  and  $z$  coordinates.

- $\iint_{\Sigma^-} = - \iint_{\Sigma}$ .
- $\iint_{\Sigma} P(x, y, z)dydz + Q(x, y, z)dzdx + R(x, y, z)dxdy = \iint_{\Sigma} (P(x, y, z) \cos \alpha + Q(x, y, z) \cos \beta + R(x, y, z) \cos \gamma).$
- If the surface of integration is symmetric:
  - If the integrand is an odd function with respect to the corresponding variable, the integral is twice the integral over half the region.
  - If the integrand is an even function, the integral is zero. (Opposite to the usual case for symmetry)

- Explanation of symmetry: Since this is a surface integral over a vector field, it involves front-back, left-right, and up-down directions, representing positive and negative contributions, respectively. Thus, if the integrand is an even function and the direction is opposite, the contributions cancel out (e.g., magnetic flux entering on one side and exiting on the other). For odd functions, the contributions add up positively because the directions are opposite, resulting in twice the integral over half the region.
- Comparison and summary: The physical meaning of surface integral over a scalar function is derived from the density function over a given spatial surface, representing the total mass of the surface. The physical meaning of surface integral over a vector field is derived from the flow rate across a given spatial surface, representing the total flux through the surface per unit time.

### 5.5.3 Computation of Surface Integrals of a Vector Field

Calculating the surface integral of a vector field is relatively complex, and it involves converting the integral into projections on different planes. Why is there a positive or negative sign involved? The reason is that the preceding  $dx dy dz$  represents the projection, which can be either positive or negative, while the latter  $dx dy dz$  represents the area, which is always positive.

- $\iint_{\Sigma} P(x, y, z) dy dz$ : Represent  $x$  using  $yz$ :

$$\Sigma : x = \phi(y, z), \quad (y, z) \in D_{yz},$$

$$\iint_{\Sigma} P(x, y, z) dy dz = \pm \iint_{D_{yz}} P[\phi(y, z), y, z] dy dz.$$

Positive for the front side, negative for the back side.

- $\iint_{\Sigma} Q(x, y, z) dz dx$ : Represent  $y$  using  $xz$ :

$$\Sigma : y = \phi(z, x), \quad (z, x) \in D_{zx},$$

$$\iint_{\Sigma} Q(x, y, z) dz dx = \pm \iint_{D_{zx}} Q[x, \phi(z, x), z] dz dx.$$

Positive for the right side, negative for the left side.

- $\oiint_{\Sigma} R(x, y, z) \, dx \, dy$ : Represent  $z$  using  $xy$ :

$$\Sigma : z = \phi(x, y), \quad (x, y) \in D_{xy},$$

$$\iint_{\Sigma} R(x, y, z) \, dx \, dy = \pm \iint_{D_{xy}} R[x, y, \phi(x, y)] \, dx \, dy.$$

Positive for the upper side, negative for the lower side.

### 5.5.4 Relationship Between Two Types of Surface Integrals

The surface integral with respect to area and the surface integral with respect to coordinates have the following conversion relationships:

$$\begin{aligned} dy \, dz &= \cos \alpha \, dS, \quad dz \, dx = \cos \beta \, dS, \quad dx \, dy = \cos \gamma \, dS, \\ \Rightarrow dS &= \frac{dy \, dz}{\cos \alpha} = \frac{dz \, dx}{\cos \beta} = \frac{dx \, dy}{\cos \gamma}, \\ \Rightarrow dy \, dz &= \frac{\cos \alpha}{\cos \beta} dz \, dx = \frac{\cos \alpha}{\cos \gamma} dx \, dy. \end{aligned}$$

Here,  $\alpha$ ,  $\beta$ , and  $\gamma$  represent the angles between the normal vector of the surface element at  $(x, y, z)$  and the  $x$ ,  $y$ , and  $z$  axes, respectively.

By using the conversion formulas above, we can not only achieve the transformation between the surface integrals of a scalar-valued function and a vector field, but also transform between different projections of surface integrals in coordinate planes. These formulas express the surface integral in terms of directional cosines along three coordinate directions, as shown below:

$$\begin{aligned} \oiint_{\Sigma} P \, dy \, dz + Q \, dz \, dx + R \, dx \, dy &= \iint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) \, dS, \\ &= \iint_{\Sigma} \left( \frac{P \cos \gamma}{\cos \alpha} + \frac{Q \cos \gamma}{\cos \beta} + R \right) dx \, dy, \\ &= \iint_{\Sigma} \left( P + \frac{Q \cos \alpha}{\cos \beta} + \frac{R \cos \alpha}{\cos \gamma} \right) dy \, dz, \\ &= \iint_{\Sigma} \left( \frac{P \cos \beta}{\cos \alpha} + Q + \frac{R \cos \beta}{\cos \gamma} \right) dz \, dx. \end{aligned}$$

The formulas above summarize the expression of the relationship between scalar-valued surface integrals and vector field surface integrals by introducing the corresponding directional cosines. The last three forms are only applicable to simple projections such as the

$XY$ -plane,  $YZ$ -plane, or  $ZX$ -plane. For more complex surfaces, it is generally more practical to directly evaluate the surface integral in coordinates, while separately considering surface segments if necessary. Thus, we only need to calculate one type of surface integral corresponding to each segment's projection.

It is important to note that the directional cosine corresponding to the normal vector of the parent surface in the conversion formulas must not be zero. Specifically,  $(\cos \alpha, \cos \beta, \cos \gamma)$  represent the directional cosines of the normal vector relative to the three coordinate axes.

## 5.6 Gauss's Theorem

The Newton-Leibniz formula transforms a definite integral into the difference of function values, and Green's theorem converts a line integral into a double integral. Gauss's theorem, in turn, transforms a surface integral into a triple integral.

### 5.6.1 Theorem Statement

**Definition:**

1. The surface of integration must be a closed surface  $\Sigma$ .
2. The normal direction of the closed surface integral must point outward.
3. In the closed region  $\Omega$  enclosed by the surface  $\Sigma$ , three functions  $P(x, y, z)$ ,  $Q(x, y, z)$ , and  $R(x, y, z)$  must have continuous first-order partial derivatives.

Under these conditions, the following relationship holds:

$$\oiint_{\Sigma} P \, dydz + Q \, dzdx + R \, dxdy = \iiint_{\Omega} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV.$$

Based on the relationship between the two types of surface integrals, we have:

$$\oiint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) \, dS = \iiint_{\Omega} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV,$$

where  $\cos \alpha$ ,  $\cos \beta$ , and  $\cos \gamma$  are the directional cosines of the normal vector relative to the  $x$ ,  $y$ , and  $z$  axes, respectively.

## 5.6.2 Computation Integrals by Gauss's Theorem

Using Gauss's theorem to calculate the surface integral of a vector field or the surface integral of a scalar-valued function involves the following basic steps:

1. Identify the functions  $P(x, y, z)$ ,  $Q(x, y, z)$ , and  $R(x, y, z)$  in the expression to be integrated. The coefficient of  $dydz$  corresponds to  $P(x, y, z)$ , the coefficient of  $dzdx$  corresponds to  $Q(x, y, z)$ , and the coefficient of  $xdy$  corresponds to  $R(x, y, z)$ . If no corresponding differential term is present, set the corresponding function to zero.
2. Ensure that the following three conditions are satisfied:
  - **Closure:** The surface must be closed.
  - **Orientation:** The normal direction must point outward.
  - **Continuity:** The functions  $P(x, y, z)$ ,  $Q(x, y, z)$ , and  $R(x, y, z)$  must have continuous partial derivatives.

If the conditions are not met, auxiliary surfaces can be added to ensure the surface is closed.

3. Compute the divergence of the vector field, which is given by:

$$\operatorname{div} \vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Then, calculate the volume integral of the divergence over the enclosed region  $\Omega$ :

$$\iiint_{\Omega} \operatorname{div} \vec{v} dV.$$

Here,  $\operatorname{div} \vec{v}$  represents the **divergence** of the vector field  $\vec{v} = (P, Q, R)$ .

4. If the closed surface satisfies the conditions of Gauss's theorem, the volume integral gives the final result. Otherwise, consider adding auxiliary surfaces to close the surface, and compute the integrals over the auxiliary surfaces. Subtract the integral over the auxiliary surfaces from the final result to ensure the correct answer.

### Example:

Compute  $I = \iint_{\Sigma} yz dzdx + 2 dx dy$ , where  $\Sigma$  is the upper half of the sphere  $x^2 + y^2 + z^2 = 4$  above the  $xOy$  plane.

Solution:

From the given integral expression, we have:

$$P = 0, \quad Q = yz, \quad R = 2, \quad \frac{\partial P}{\partial x} = 0, \quad \frac{\partial Q}{\partial y} = z, \quad \frac{\partial R}{\partial z} = 0.$$

Since  $\Sigma$  represents only the upper half of the sphere, we need to add the bottom surface  $\Sigma_0$  (i.e.,  $z = 0$  and  $x^2 + y^2 \leq 4$ ) to form a complete closed hemisphere. The normal vector of  $\Sigma_0$  points in the negative  $z$ -direction. Thus, the integral becomes:

$$I = \oiint_{\Sigma+\Sigma_0} - \iint_{\Sigma_0}.$$

Applying Gauss's theorem:

$$\oiint_{\Sigma+\Sigma_0} = \iiint_{\Omega} z \, dV = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\psi \int_0^2 r \cos \psi \cdot r^2 \sin \psi \, dr.$$

Evaluating the integral:

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin \psi \, d(\sin \psi) \int_0^2 r^3 \, dr = 2\pi \times \frac{1}{2} \sin^2 \psi \Big|_0^{\frac{\pi}{2}} \times \frac{r^4}{4} \Big|_0^2 = 4\pi.$$

The integral over the bottom surface  $\Sigma_0$  is:

$$\iint_{\Sigma_0} yz \, dzdx + 2 \, dxdy = \iint_{\Sigma_0} 2 \, dxdy = -2 \iint_{D_{xy}} dxdy = -8\pi.$$

Therefore, the final result is:

$$I = 4\pi + 8\pi = 12\pi.$$

## 5.7 Stokes' Theorem

### 5.7.1 Theorem Statement

1. Functions  $P(x, y, z)$ ,  $Q(x, y, z)$ , and  $R(x, y, z)$  have continuous first-order partial derivatives within a closed region  $\Omega$  in space.
2.  $\Gamma$  is a smooth, closed, oriented curve in  $\Omega$ , and  $\Sigma$  is a smooth oriented surface in  $\Omega$  whose boundary is  $\Gamma$ .

3. The orientation of  $\Gamma$  and the normal orientation of  $\Sigma$  must follow the **right-hand rule** (i.e., when the four fingers of the right hand point along the direction of the curve  $\Gamma$ , the thumb should point in the direction of the normal vector to the surface  $\Sigma$ ).

Then, the following equation holds:

$$\oint_{\Gamma} P dx + Q dy + R dz = \iint_{\Sigma} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

Alternatively, using directional cosines:

$$\oint_{\Gamma} P dx + Q dy + R dz = \iint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds,$$

where  $(\cos \alpha, \cos \beta, \cos \gamma)$  are the direction cosines of the unit normal vector to the surface  $\Sigma$ , and  $(\cos u, \cos v, \cos w)$  are the direction cosines of the tangent vector to the curve  $\Gamma$ .

### 5.7.2 Computation of Integrals by Stokes' Theorem

1. Determine whether the integral curve is a closed curve. If not, consider adding a simple auxiliary curve to form a closed curve. Generally, the direction of the added curve should be consistent with the direction of the original curve.
2. Select a simple surface bounded by the closed curve as the integration surface. The direction is determined by the Right-Hand Rule (i.e., when the four fingers of the right hand point in the direction of the curve, the direction pointed by the thumb should be taken as the surface direction). The direction of the selected surface is not unique, and simple surfaces, such as planes or spheres, are generally preferred.
3. Ensure that the three conditions of Stokes' theorem are satisfied (the same as the ones of Gauss's theorem, i.e., close, orientation, and continuity of partial derivatives). Then, use Stokes' theorem to transform the curve integral into a surface integral.
4. Apply the appropriate method for calculating the surface integral to obtain the result.

#### Example:

Given a curve  $L$  on the surface  $\Sigma : 4x^2 + y^2 + z^2 = 1$ , with boundaries  $x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$ , and the surface  $\Sigma$  oriented upward. The direction of the curve  $L$  and the orientation of the surface  $\Sigma$  satisfy the right-hand rule. Compute the integral:

$$I = \oint_L (y^2 - \cos z) dx + 2x^2 dy + (2xyz + x \sin z) dz.$$

Solution:

Since the direction of the curve  $L$  and the orientation of the surface  $\Sigma$  satisfy the right-hand rule, by Stokes' Theorem, we have:

$$I = \iint_{\Sigma} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy,$$

where  $P = y^2 - \cos z$ ,  $Q = 2x^2$ , and  $R = 2xyz + x \sin z$ . Therefore, we compute the partial derivatives and the integral becomes:

$$I = \iint_{\Sigma} (-2xz) dydz + 0 \cdot dzdx + z^2 dxdy.$$

We add auxiliary planes:

- $\Sigma_1$ :  $z = 0$  with  $4x^2 + y^2 \leq 1$  (bottom plane),
- $\Sigma_2$ :  $y = 0$  with  $4x^2 + z^2 \leq 1$  (back plane),
- $\Sigma_3$ :  $x = 0$  with  $y^2 + z^2 \leq 1$  (left plane).

The surfaces  $\Sigma$ ,  $\Sigma_1$ ,  $\Sigma_2$ , and  $\Sigma_3$  form a closed region  $\Omega$ , with the normal vector pointing outward.

By Stokes' Theorem:

$$\oiint_{\Sigma+\Sigma_1+\Sigma_2+\Sigma_3} (-2xz) dydz + z^2 dxdy = \iiint_{\Omega} (-2z + 2z) dv = 0.$$

Since  $\Sigma_2$  and  $\Sigma_3$  are perpendicular to the  $yz$ -plane, their integrals with respect to  $x$  and  $y$  are zero. Hence:

$$\iint_{\Sigma+\Sigma_1+\Sigma_2+\Sigma_3} (-2xz) dydz + z^2 dxdy = \iint_{\Sigma_1} (-2xz) dydz + z^2 dxdy = 0.$$

Substituting the integrals for  $\Sigma_1$  and  $\Sigma_2$ , we get:

$$\iint_{\Sigma_1} (-2xz) dydz + z^2 dxdy = 0,$$

which implies:

$$I = \oiint_{\Sigma+\Sigma_1+\Sigma_2+\Sigma_3} (-2xz) dydz + z^2 dxdy - \iint_{\Sigma_1+\Sigma_2+\Sigma_3} (-2xz) dydz + z^2 dxdy = 0 - 0 = 0.$$

Therefore, the final result is:

$$I = 0.$$

### 5.7.3 Physical Interpretation of Stokes' Theorem

Given a vector field  $\vec{A} = (P, Q, R)$ , Stokes' theorem can be expressed as:

$$\oint_{\Gamma} \vec{A}_{\vec{\tau}} ds = \iint_{\Sigma} (\text{rot } \vec{A})_{\vec{n}} dS$$

This equation states that the circulation of the vector field along the curve  $\Gamma$  (left-hand side) is equal to the flux of the curl of the vector field through the surface  $\Sigma$  (right-hand side). Here,  $\vec{n}$  represents the unit normal vector to the surface  $\Sigma$  (typically directly taken as the unit vector), and  $\vec{\tau}$  represents the tangent vector along the curve  $\Gamma$  (typically directly taken as the unit vector).

### 5.7.4 Equivalent Descriptions of Path Independence

Let  $G$  be a simply connected domain in space, and functions  $P(x, y, z)$ ,  $Q(x, y, z)$ , and  $R(x, y, z)$  have continuous first-order partial derivatives in  $G$ . Then, the following four descriptions are equivalent:

1. For any closed smooth curve  $\Gamma$  in  $G$ ,

$$\oint_{\Gamma} P dx + Q dy + R dz = 0.$$

2. For any smooth curve  $\Gamma$  in  $G$ , the integral

$$\int_{\Gamma} P dx + Q dy + R dz$$

is independent of the path.

3. There exists a scalar function  $u(x, y, z)$  in  $G$  such that

$$du(x, y, z) = P dx + Q dy + R dz.$$

4. At every point in  $G$ , the following equalities hold:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y},$$

or equivalently,

$$\nabla \cdot \vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0,$$

where  $\vec{v} = (P, Q, R)$ .

**Note:** When  $\nabla \cdot \vec{v} = 0$ , the integral is path-independent, meaning that one can freely choose any convenient path to calculate the curve integral with respect to coordinates. Moreover, if the integral can be expressed in terms of a potential function  $u$ , we have

$$\int_{\Gamma:A \rightarrow B} P dx + Q dy + R dz = u(B) - u(A),$$

where  $du = P dx + Q dy + R dz$ , and  $\vec{v} = (P, Q, R)$ .

### 5.7.5 Several Key Concepts

- **Hamilton Operator**

The Hamilton operator  $\nabla$  is defined as:

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

- **Gradient (Vector)**

For a scalar function  $u = u(x, y, z)$ :

$$\nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

- **Divergence (Scalar)**

For a vector  $\vec{A} = (P, Q, R)$  composed of three scalar functions:

$$\nabla \cdot \vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- **Curl (Vector)**

For a vector  $\vec{A} = (P, Q, R)$  composed of three scalar functions:

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

The curl is also denoted as  $\text{curl } \vec{A}$ .

- **Directional Derivative**

For a differentiable function  $f(x, y, z)$  in the direction  $\vec{u} = (\cos \alpha, \cos \beta, \cos \gamma)$ :

$$\frac{\partial f}{\partial \vec{u}} = D_{\vec{u}} f(x, y, z) = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta + \frac{\partial f}{\partial z} \cos \gamma$$

# Appendix: Calculus Formulas

## 1. Differentiation Formulas

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \cdot \tan x$$

$$(\csc x)' = -\csc x \cdot \cot x$$

$$(a^x)' = a^x \ln a$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

## 2. Integration Formulas

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = -\ln |\cos x| + C$$

$$\int \cot x \, dx = \ln |\sin x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln |\csc x - \cot x| + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left( x + \sqrt{x^2 \pm a^2} \right) + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} I_{n-2}$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left( x + \sqrt{x^2 + a^2} \right) + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left( x + \sqrt{x^2 - a^2} \right) + C$$

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$$

### 3. Trigonometric Identities

#### Double-Angle Formulas

$$\begin{aligned}\sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \\ \tan(2x) &= \frac{2 \tan x}{1 - \tan^2 x}\end{aligned}$$

#### Half-Angle Formulas

$$\begin{aligned}\sin\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1 - \cos x}{2}} \\ \cos\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1 + \cos x}{2}} \\ \tan\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}\end{aligned}$$

#### Product-to-Sum Formulas

$$\begin{aligned}\sin A \pm \sin B &= 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \\ \cos A + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\ \cos A - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right)\end{aligned}$$

#### Sum-to-Product Formulas

$$\begin{aligned}\sin A \sin B &= \frac{1}{2}[\cos(A - B) - \cos(A + B)] \\ \cos A \cos B &= \frac{1}{2}[\cos(A + B) + \cos(A - B)] \\ \sin A \cos B &= \frac{1}{2}[\sin(A + B) + \sin(A - B)]\end{aligned}$$

#### Integral Transformations by Trigonometric Substitution

The following formulas serve as a reference for solving integrals involving trigonometric identities and hyperbolic functions.

$$\sin x = \frac{2u}{1 + u^2}, \quad \cos x = \frac{1 - u^2}{1 + u^2}, \quad u = \tan \frac{x}{2}, \quad dx = \frac{2 du}{1 + u^2}$$

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